GMAC Bank

Student Loans
for up to
$40,000 per year *

Defer payments until after graduation.**
Fast preliminary approval, usually in minutes.

Apply Now!
Go to gmacbankfunding.com

Apply online in as little as 15 minutes

- Loans up to $40,000 per academic year*
- Good for tuition and other educational expenses: books, fees, a laptop, room and board, travel home, etc.
- Get a check in as few as 5 business days
- Start payments now or up to six months after graduation**
- Reduce your interest rate by as much as 0.50% with automatic payments***

All loans are subject to application and credit approval.

* Undergraduate and graduate borrowers may borrow annually up to the lesser of the cost of attendance or $30,000 ($40,000 for certain schools where it has been determined that the annual cost of attendance exceeds $30,000). Borrowers in the Continuing Education loan program may borrow annually up to $30,000.

** Undergraduate students may choose to defer repayment until six months after graduation or ceasing to be enrolled at least half time in school. Interest only and immediate repayment options also available.

*** A 0.25% interest rate reduction is available for borrowers who elect to have monthly principal and interest payments transferred electronically from a savings or checking account. The interest rate reduction will begin when automatic principal and interest payments start, and will remain in effect as long as automatic payments continue without interruption. This reduced interest rate will return to contract rate if automatic payments are cancelled, rejected or returned for any reason. Upon request, borrowers are also entitled to an additional 0.25% interest rate reduction if (1) the first 36 payments of principal and interest are paid on time, and (2) at any time prior to the 36th on time payment, the borrower who receives the monthly bill elects to have monthly principal and interest payments transferred electronically from a savings or checking account, and continues to make such automatic payments through the 36th payment. This reduced interest rate will not be returned to contract rate if, after receiving the benefit, the borrower discontinues automatic electronic payment. The lender and servicer reserve the right to modify or discontinue borrower benefit programs (other than the co-signer release benefit) at any time without notice.

Chapter 7

Exercise 7–1

1. $\dfrac{\sin\theta}{\tan\theta}$
$\dfrac{\sin\theta}{\frac{\sin\theta}{\cos\theta}}$
$\sin\theta\cdot\dfrac{\cos\theta}{\sin\theta}$
$\cos\theta$

5. $\cot^2\theta\sin^2\theta$
$\dfrac{\cos^2\theta}{\sin^2\theta}\cdot\sin^2\theta$
$\cos^2\theta$

9. $\dfrac{(\sec\theta-1)(\sec\theta+1)}{\sin^2\theta}$
$\dfrac{\sec^2\theta-1}{\sin^2\theta}$
$\dfrac{\tan^2\theta}{\sin^2\theta}$
$\tan^2\theta+1=\sec^2\theta$, so $\tan^2\theta=\sec^2\theta-1$
$\tan^2\theta\,\dfrac{1}{\sin^2\theta}$
$\dfrac{\sin^2\theta}{\cos^2\theta}\cdot\dfrac{1}{\sin^2\theta}$
$\dfrac{1}{\cos^2\theta}$
$\sec^2\theta$

13. $\sec x-\tan x\sin x$
$\sec x-\dfrac{\sin x}{\cos x}\cdot\sin x$
$\dfrac{1}{\cos x}-\dfrac{\sin^2x}{\cos x}$
$\dfrac{1-\sin^2x}{\cos x}$
$\sin^2\theta+\cos^2\theta=1$, so $\cos^2\theta=1-\sin^2\theta$
$\dfrac{\cos^2x}{\cos x}$
$\cos x$

17. $\dfrac{\csc^2\theta-1}{\csc^2\theta}$
$\dfrac{\cot^2\theta}{\csc^2\theta}$
$\dfrac{\cos^2\theta}{\sin^2\theta}\cdot\sin^2\theta$
$\cos^2\theta$

21. $\dfrac{\csc\theta\sin\theta}{\cot\theta}$
$\csc\theta\sin\theta\tan\theta$
$\dfrac{1}{\sin\theta}\cdot\sin\theta\cdot\tan\theta$
$\tan\theta$

25. $\cos^2\theta(1+\cot^2\theta)$
$\cos^2\theta\csc^2\theta$
$\dfrac{\cos^2\theta}{\sin^2\theta}$
$\cot^2\theta$

29. $\tan^2\theta-\sec^2\theta$
$\tan^2\theta-(\tan^2\theta+1)$
-1

33. $\csc\theta+\cot\theta$
$\dfrac{1}{\sin\theta}+\dfrac{\cos\theta}{\sin\theta}$
$\dfrac{1+\cos\theta}{\sin\theta}$

37. $\dfrac{1+\csc\theta}{1+\sec\theta}$
$\dfrac{1+\frac{1}{\sin\theta}}{1+\frac{1}{\cos\theta}}$
$\dfrac{\frac{\sin\theta+1}{\sin\theta}}{\frac{\cos\theta+1}{\cos\theta}}$
$\dfrac{\sin\theta+1}{\sin\theta}\cdot\dfrac{\cos\theta}{\cos\theta+1}$
$\dfrac{\cos\theta}{\sin\theta}\cdot\dfrac{\sin\theta+1}{\cos\theta+1}$
$\cot\theta\left(\dfrac{1+\sin\theta}{1+\cos\theta}\right)$

41. $\dfrac{1+\cot^2\theta}{\tan^2\theta}$
$\dfrac{1}{\tan^2\theta}+\dfrac{\cot^2\theta}{\tan^2\theta}$
$\cot^2\theta+\cot^2\theta\cot^2\theta$
$\cot^2\theta(1+\cot^2\theta)$
$\cot^2\theta\csc^2\theta$

45. $\dfrac{\cot^2\theta}{\csc\theta+1}$
$\dfrac{\csc^2\theta-1}{\csc\theta+1}$
$\cot^2\theta+1=\csc^2\theta$, so $\cot^2\theta=\csc^2\theta-1$
$\dfrac{(\csc\theta-1)(\csc\theta+1)}{\csc\theta+1}$
$x^2-1=(x-1)(x+1)$.
$\csc\theta-1$

49. $2\cos^2x-1$
$2\cos^2x-(\sin^2x+\cos^2x)$
$\cos^2x-\sin^2x$

53. $\dfrac{\cot x+1}{\cot x-1}$
$\dfrac{\frac{\cos x}{\sin x}+1}{\frac{\cos x}{\sin x}-1}$
$\dfrac{\frac{\cos x}{\sin x}+\frac{\sin x}{\sin x}}{\frac{\cos x}{\sin x}-\frac{\sin x}{\sin x}}$
$\dfrac{\frac{\cos x}{\sin x}+\frac{\sin x}{\sin x}}{\frac{\cos x}{\sin x}-\frac{\sin x}{\sin x}}\cdot\dfrac{\sin x}{\sin x}$
$\dfrac{\cos x+\sin x}{\cos x-\sin x}$

57. $\dfrac{1}{1-\sin x}+\dfrac{1}{1+\sin x}$
$\dfrac{(1+\sin x)+(1-\sin x)}{1-\sin^2x}$
$\dfrac{a}{b}+\dfrac{c}{d}=\dfrac{ad+bc}{bd}$
$\dfrac{2}{\cos^2x}$
$2\sec^2x$

61. $\csc^2y+\sec^2y$
$\dfrac{1}{\sin^2y}+\dfrac{1}{\cos^2y}$
$\dfrac{\cos^2y+\sin^2y}{\sin^2y\cos^2y}$
$\dfrac{1}{\sin^2y\cos^2y}$
$\csc^2y\sec^2y$

65. $\sec^4x-\sec^2x$
$\sec^2x(\sec^2x-1)$
$(\tan^2x+1)[(\tan^2x+1)-1)$
$(\tan^2x+1)(\tan^2x)$
$\tan^4x+\tan^2x$

In each of problems 67 – 76, let $\theta=\frac{\pi}{3}$; $\sin\frac{\pi}{3}=\frac{\sqrt3}{2}$, $\cos\frac{\pi}{3}=\frac{1}{2}$; $\tan\frac{\pi}{3}=\sqrt3$.

69. $\sec\theta=\dfrac{1}{\csc\theta}$
$2=\dfrac{1}{\frac{2}{\sqrt3}}$
$2\neq\frac{\sqrt3}{2}$

73. $\csc\theta+\sec\theta\cot\theta=2$
$\frac{2}{\sqrt3}+2\cdot\frac{1}{\sqrt3}=2$
$\frac{4}{\sqrt3}\neq2$

77. (a) $(\csc^2\theta-1)(\sec^2\theta-1)=1$
$\theta=\frac{\pi}{6}$: $(\csc^2\frac{\pi}{6}-1)(\sec^2\frac{\pi}{6}-1)=1$
$(2^2-1)\left(\left(\frac{2}{\sqrt3}\right)^2-1\right)$
$(3)\left(\frac{4}{3}-1\right)$
1

$\theta=\frac{\pi}{4}$: $(\csc^2\frac{\pi}{4}-1)(\sec^2\frac{\pi}{4}-1)=1$
$((\sqrt2)^2-1)((\sqrt2)^2-1)$
$(2-1)(2-1)$
1

(b) Yes: $(\csc^2\theta-1)(\sec^2\theta-1)=1$
$\cot^2\theta\tan^2\theta$
$\dfrac{1}{\tan^2\theta}\tan^2\theta$
1

Exercise 7–2

1. $\sin18°=\cos(90°-18°)=\cos72°$

5. $\sec\frac{\pi}{3}=\csc(\frac{\pi}{2}-\frac{\pi}{3})=\csc\frac{\pi}{6}$

9. $\sec(-\frac{3\pi}{4})=\csc\left(\frac{\pi}{2}-(-\frac{3\pi}{4})\right)=\csc\frac{5\pi}{4}$

13. $\cos20°\csc70°=\cos20°\sec20°=\cos20°\cdot\dfrac{1}{\cos20°}=1$

17. $\tan^2 8°-\csc^2 82°=\tan^2 8°-\sec^2 8°$
$=-1$ (Since $\tan^2\theta+1=\sec^2\theta$, $\tan^2\theta-\sec^2\theta=-1$.)

21. $\sec\frac{\pi}{6}\sin\frac{\pi}{3}=\csc\frac{\pi}{3}\sin\frac{\pi}{3}=1$

25. $\sec^2\frac{\pi}{3}-\cot^2\frac{\pi}{6}$
$\sec^2\frac{\pi}{3}-\tan^2\frac{\pi}{3}=1$ (Since $\tan^2\theta+1=\sec^2\theta$, $\sec^2\theta-\tan^2\theta=1$.)

29. $\sin\frac{5\pi}{12}=\sin\left(\frac{\pi}{4}+\frac{\pi}{6}\right)$ $=\sin\frac{\pi}{4}\cos\frac{\pi}{6}+\cos\frac{\pi}{4}\sin\frac{\pi}{6}$
$=\frac{\sqrt2}{2}\cdot\frac{\sqrt3}{2}+\frac{\sqrt2}{2}\cdot\frac{1}{2}=\frac{\sqrt6+\sqrt2}{4}$

$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

33. $\sin 15° = \sin(45° - 30°) \quad = \sin 45° \cos 30° - \cos 45° \sin 30°$ $\qquad \sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

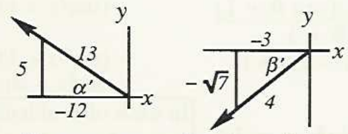
$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

37. $\tan 75° = \tan(45° + 30°) \quad = \dfrac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°}$ $\qquad \tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$$= \frac{1 + \frac{\sqrt{3}}{3}}{1 - 1\left(\frac{\sqrt{3}}{3}\right)}\cdot\frac{3}{3} = \frac{3+\sqrt{3}}{3-\sqrt{3}}\cdot\frac{3+\sqrt{3}}{3+\sqrt{3}} = \frac{12+6\sqrt{3}}{6} = 2+\sqrt{3}$$

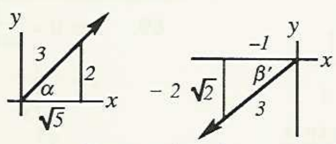
41. $\tan(\alpha - \beta) \ = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

$$= \frac{-\frac{5}{12} - \frac{\sqrt{7}}{3}}{1 + \left(-\frac{5}{12}\right)\left(\frac{\sqrt{7}}{3}\right)} = \frac{\frac{-15-12\sqrt{7}}{36}}{1 - \frac{5\sqrt{7}}{36}}\cdot\frac{36}{36} = \frac{-15\ -12\sqrt{7}}{36 - 5\sqrt{7}}$$



45. $\cos(\alpha - \beta) \ = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

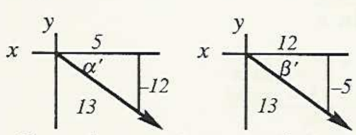
$$= \frac{\sqrt{5}}{3}\left(-\frac{1}{3}\right) + \frac{2}{3}\left(-\frac{2\sqrt{2}}{3}\right) = \frac{-\sqrt{5} - 4\sqrt{2}}{9}$$



49. $\tan(\alpha - \beta) \ = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

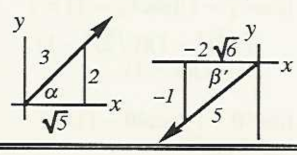
$$= \frac{-\frac{12}{5} - \left(-\frac{5}{12}\right)}{1 + \left(-\frac{12}{5}\right)\left(-\frac{5}{12}\right)} = \frac{-\frac{12}{5} + \frac{5}{12}}{1 + 1} = \frac{\frac{-144+25}{60}}{2}$$

$$= \frac{1}{2}\left(-\frac{119}{60}\right) = -\frac{119}{120}.$$



53. $\sin(\alpha - \beta) \ = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$$= \frac{2}{3}\left(-\frac{2\sqrt{6}}{5}\right) - \frac{\sqrt{5}}{3}\left(-\frac{1}{5}\right) = \frac{-4\sqrt{6}+\sqrt{5}}{15}$$



57. $\sin(\pi - \theta) \ = \sin\pi\cos\theta - \cos\pi\sin\theta$

$= 0\cos\theta - (-1)\sin\theta$

$= \sin\theta$

61. $\tan(\pi - \theta) \ = \dfrac{\tan\pi - \tan\theta}{1 + \tan\pi\tan\theta}$

$= \dfrac{0 - \tan\theta}{1 + 0\tan\theta}$

$= -\tan\theta$

65. $\tan(\theta + \pi) \ = \dfrac{\tan\theta + \tan\pi}{1 - \tan\theta\tan\pi}$

$= \dfrac{\tan\theta + 0}{1 - \tan\theta\,(0)}$

$= \tan\theta$

69. $\frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$

$\frac{1}{2}[\cos\alpha\cos\beta + \sin\alpha\sin\beta - (\cos\alpha\cos\beta - \sin\alpha\sin\beta)]$

$\frac{1}{2}[2\sin\alpha\sin\beta] = \sin\alpha\sin\beta$

73. $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

This was shown true in the text.

$\cos(\alpha + (-\beta)) = \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta)$

Replace β by $-\beta$. This is valid since the identity is true for all angles and α.

$\cos(\alpha - \beta) = \cos\alpha\cos\beta - \sin\alpha\,[-\sin\beta]$

$\alpha + (-\beta) = \alpha - \beta$; $\cos(-\theta) = \cos\theta$; $\sin(-\theta) = -\sin\theta$.

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.

This statement is true since the preceeding statements are true.

77. $\sin(\alpha + \beta) \quad = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ Identity [3].

$\sin(\alpha + (-\beta)) = \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta)$

$\sin(\alpha - \beta) \quad = \sin\alpha\cos\beta + \cos\alpha\,[-\sin\beta]$

Cosine is an even function, sine is odd.

$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$

This is identity [4].

81. $\csc\left(\frac{\pi}{2} - \theta\right) = \dfrac{1}{\sin\left(\frac{\pi}{2} - \theta\right)} = \dfrac{1}{\cos\theta} = \sec\theta.$

Exercise 7–3

1. $2\sin\frac{\pi}{4}\cos\frac{\pi}{4}$

$2\sin\alpha\cos\alpha = \sin 2\alpha$

Let $\alpha = \frac{\pi}{4}$, so $2\alpha = \frac{\pi}{2}$, and $\sin 2\alpha$ is $\sin\frac{\pi}{2}$.

5. $1 - 2\sin^2\frac{\pi}{10}$

$1 - 2\sin^2\alpha = \cos 2\alpha$

Let $\alpha = \frac{\pi}{10}$, so $2\alpha = \frac{\pi}{5}$, so $\cos 2\alpha = \cos\frac{\pi}{5}$.

9. $2\sin 6\theta\cos 6\theta$

$2\sin\alpha\cos\alpha = \sin 2\alpha$ Let $\alpha = 6\theta$

$\sin 2\alpha$ becomes $\sin 12\theta$.

13. $\dfrac{10\tan 3\theta}{1 - \tan^2 3\theta}$

$\dfrac{2\tan\alpha}{1 - \tan^2\alpha} = \tan 2\alpha$ Identity

$\dfrac{10\tan\alpha}{1 - \tan^2\alpha} = 5\tan 2\alpha$ Multiply each member by 5

Let $\alpha = 3\theta$, so $2\alpha = 6\theta$. Thus $5\tan 2\alpha$ is $5\tan 6\theta$.

17. $3\cos^2 3\theta - 3\sin^2 3\theta$

$\cos^2\alpha - \sin^2\alpha = \cos 2\alpha$ Identity

$3\cos^2\alpha - 3\sin^2\alpha = 3\cos 2\alpha$

Multiply each member by 3. α represents 3θ, so 2α is 6θ. Then $3\cos 2\alpha$ means $3\cos 6\theta$.

21. $\cos\frac{5\pi}{6} = \cos^2\theta - \sin^2\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$2\theta = \frac{5\pi}{6}$, so $\theta = \frac{5\pi}{12}$.

$\cos\frac{5\pi}{6} = \cos^2\frac{5\pi}{12} - \sin^2\frac{5\pi}{12}$.

25. $\sin 10° = \sqrt{\dfrac{1 - \cos\theta}{2}}$

$\sin\frac{\alpha}{2} = \sqrt{\dfrac{1 - \cos\alpha}{2}}$

$\frac{\alpha}{2} = 10°$, so $\alpha = \theta = 20°$.

$$\sin 10° = \sqrt{\frac{1 - \cos 20°}{2}}$$

29. $\tan \frac{2\pi}{5} = \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}}$

$$\tan \frac{\alpha}{2} = \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}}$$

$\frac{\alpha}{2} = \frac{2\pi}{5}$, so $\alpha = \theta = \frac{4\pi}{5}$, and

$$\tan \frac{2\pi}{5} = \sqrt{\frac{1 - \cos \frac{4\pi}{5}}{1 + \cos \frac{4\pi}{5}}}.$$

33. $\cos \theta = -\frac{4}{5}, \frac{\pi}{2} < \theta < \pi$

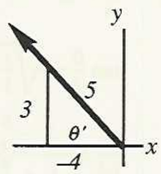
$\sin 2\theta = 2 \sin \theta \cos \theta = 2 \cdot \frac{3}{5}\left(-\frac{4}{5}\right)$

$= -\frac{24}{25}$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= \left(-\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{7}{25}$

$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = -\frac{24}{7}$



37. $\sec \theta = -\frac{5}{2},\ \pi < \theta < \frac{3\pi}{2}$: $\cos \theta = -\frac{2}{5}$

$\frac{\pi}{2} \le \frac{\theta}{2} \le \frac{3\pi}{4}$, ($\frac{\theta}{2}$ in qII) so $\sin \frac{\theta}{2} > 0$, $\cos \frac{\theta}{2} < 0$, $\tan \frac{\theta}{2} < 0$.

$$\sin \frac{\theta}{2} = \sqrt{\frac{1 - \cos \theta}{2}} = \sqrt{\frac{1 - \left(-\frac{2}{5}\right)}{2}} = \sqrt{\frac{1}{2} \cdot \frac{7}{5}}$$

$$= \frac{\sqrt{7}}{\sqrt{10}} = \frac{\sqrt{70}}{10}$$

$$\cos \frac{\theta}{2} = -\sqrt{\frac{1 + \cos \theta}{2}} = -\sqrt{\frac{1 + \left(-\frac{2}{5}\right)}{2}} = -\sqrt{\frac{1}{2} \cdot \frac{3}{5}}$$

$$= -\frac{\sqrt{3}}{\sqrt{10}} = -\frac{\sqrt{30}}{10}$$

$$\tan \frac{\theta}{2} = -\sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}} = -\sqrt{\frac{1-\left(-\frac{2}{5}\right)}{1+\left(-\frac{2}{5}\right)}}$$

$$= -\sqrt{\frac{7}{10} \div \frac{3}{10}} = -\sqrt{\frac{7}{3}} = -\frac{\sqrt{7}}{\sqrt{3}} = -\frac{\sqrt{21}}{3}$$

41. 15°, or $\frac{\pi}{12}$

$$\sin 15° = \sqrt{\frac{1 - \cos 30°}{2}} = \sqrt{\frac{1}{2}\left(1 - \frac{\sqrt{3}}{2}\right)} = \sqrt{\frac{2 - \sqrt{3}}{4}} = \frac{\sqrt{2 - \sqrt{3}}}{2}$$

$$\cos 15° = \sqrt{\frac{1 + \cos 30°}{2}} = \sqrt{\frac{1}{2}\left(1 + \frac{\sqrt{3}}{2}\right)} = \sqrt{\frac{2 + \sqrt{3}}{4}} = \frac{\sqrt{2 + \sqrt{3}}}{2}$$

$$\tan 15° = \frac{1 - \cos 30°}{\sin 30°} = \frac{1 - \frac{\sqrt{3}}{2}}{\frac{1}{2}} \frac{2}{2} = \frac{2 - \sqrt{3}}{1} = 2 - \sqrt{3}$$

45. $\cos 37.5° = \cos(15° + 22.5°) = \cos 15° \cos 22.5° - \sin 15° \sin 22.5°$

$$= \frac{\sqrt{2+\sqrt{3}}}{2} \bullet \frac{\sqrt{2+\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{3}}}{2} \frac{\sqrt{2-\sqrt{2}}}{2} = \frac{\sqrt{4+\sqrt{6}+2\sqrt{3}+2\sqrt{2}} - \sqrt{4+\sqrt{6}-2\sqrt{3}-2\sqrt{2}}}{4}$$

Note: A simpler solution is as follows: Use $\cos 75° = \cos(30° + 45°)$ and expand. This gives $\cos 75° = \frac{\sqrt{6} - \sqrt{2}}{4}$. Use this value in the identity $\cos \frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos\alpha}{2}}$, with $\alpha = 75°$. This produces the solution $\frac{1}{4}\sqrt{2\sqrt{6} - 2\sqrt{2} + 8}$.

49. $\sin 2\theta + 1$

$2 \sin \theta \cos \theta + 1$

$2 \sin \theta \cos \theta + \sin^2\theta + \cos^2\theta$

$\sin^2\theta + 2 \sin \theta \cos \theta + \cos^2\theta$

$(\sin \theta + \cos \theta)^2$

53. $\frac{1 + \cos 2\theta}{1 - \cos 2\theta}$

$\frac{1 + (2 \cos^2\theta - 1)}{1 - (1 - 2 \sin^2\theta)}$

$\frac{2 \cos^2\theta}{2 \sin^2\theta}$

$\frac{\cos^2\theta}{\sin^2\theta}$

$\cot^2\theta$

57. $\sin 2\theta - 4 \sin^3\theta \cos \theta$

$2 \sin \theta \cos \theta - 4 \sin^3\theta \cos \theta$

$2 \sin \theta \cos \theta(1 - 2 \sin^2\theta)$

$\sin 2\theta \cos 2\theta$

61. $\tan 2\theta \quad = \frac{2(\tan \theta + \tan^3\theta)}{1 - \tan^4\theta}$

$\frac{2 \tan \theta}{1 - \tan^2\theta} = \frac{2 \tan \theta(1 + \tan^2\theta)}{(1 - \tan^2\theta)(1 + \tan^2\theta)}$

$= \frac{2 \tan \theta}{1 - \tan^2\theta}$

65. $\frac{1 - \tan^2\theta}{1 + \tan^2\theta}$

$\frac{1 - \frac{\sin^2\theta}{\cos^2\theta}}{1 + \frac{\sin^2\theta}{\cos^2\theta}}$

$\frac{\frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta}}{\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}}$

$\frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + \sin^2\theta}$

$\frac{\cos 2\theta}{1}$

$\cos 2\theta$

69. $\cos^2\frac{\theta}{2}$

$\left(\sqrt{\frac{1 + \cos \theta}{2}}\right)^2$

$\frac{1 + \cos \theta}{2}$

$\frac{1 + \cos \theta}{2} \cdot \frac{1 - \cos \theta}{1 - \cos \theta}$

$\frac{1 - \cos^2\theta}{2 - 2 \cos \theta}$

73. $\sin^2\frac{\theta}{2} - \cos^2\frac{\theta}{2} = -\cos \theta$

$\frac{1 - \cos \theta}{2} - \frac{1 + \cos \theta}{2}$

$\frac{-2 \cos \theta}{2}$

$-\cos \theta$

77. $4 \sin^2\frac{\theta}{2} \cos^2\frac{\theta}{2}$

$4\left(\sqrt{\frac{1 - \cos \theta}{2}}\right)^2\left(\sqrt{\frac{1 + \cos \theta}{2}}\right)^2$

$4\ \frac{1 - \cos \theta}{2} \cdot \frac{1 + \cos \theta}{2}$

$4\ \frac{1 - \cos^2\theta}{4}$

$\sin^2\theta$

81. $\sin 3\theta$

$\sin(2\theta + \theta)$

$\sin 2\theta \cos \theta + \cos 2\theta \sin \theta$

$(2 \sin \theta \cos \theta)(\cos \theta)$

$+ (1 - 2 \sin^2\theta)(\sin \theta)$

$2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta$ $\qquad$ $2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta$

$2\sin\theta(1 - \sin^2\theta) + \sin\theta - 2\sin^3\theta$ $\qquad$ $3\sin\theta - 4\sin^3\theta$

85. $\frac{3}{16}a\left(\frac{1-\cos 2\alpha}{1-\cos\alpha}\right) = \frac{3}{16}a\frac{1-(\cos^2\alpha - \sin^2\alpha)}{1-\cos\alpha} = \frac{3}{16}a\frac{(\sin^2\alpha + \cos^2\alpha) - (\cos^2\alpha - \sin^2\alpha)}{1-\cos\alpha}$

$= \frac{3}{16}a\frac{2\sin^2\alpha}{1-\cos\alpha} = \frac{3}{8}a\frac{1-\cos^2\alpha}{1-\cos\alpha} = \frac{3}{8}a\frac{(1-\cos\alpha)(1+\cos\alpha)}{1-\cos\alpha} = \frac{3}{8}a\,(1+\cos\alpha)$

89. (a) Problem 41 shows that $\tan 15° = 2 - \sqrt{3}$.

(b) Problem 41 shows that $\sin 15° = \frac{\sqrt{2-\sqrt{3}}}{2}$, and that

$\cos 15° = \frac{\sqrt{2+\sqrt{3}}}{2}$, so $\tan 15° = \frac{\frac{\sqrt{2-\sqrt{3}}}{2}}{\frac{\sqrt{2+\sqrt{3}}}{2}}$.

(c) $\frac{\frac{\sqrt{2-\sqrt{3}}}{2}}{\frac{\sqrt{2+\sqrt{3}}}{2}} = \frac{\sqrt{2-\sqrt{3}}}{2}\cdot\frac{2}{\sqrt{2+\sqrt{3}}} = \frac{\sqrt{2-\sqrt{3}}}{\sqrt{2+\sqrt{3}}}$

$= \frac{\sqrt{2-\sqrt{3}}}{\sqrt{2+\sqrt{3}}}\cdot\frac{\sqrt{2+\sqrt{3}}}{\sqrt{2+\sqrt{3}}} = \frac{\sqrt{(2-\sqrt{3})(2+\sqrt{3})}}{2+\sqrt{3}}$

$= \frac{1}{2+\sqrt{3}} = \frac{1}{2+\sqrt{3}}\cdot\frac{2-\sqrt{3}}{2-\sqrt{3}} = \frac{2-\sqrt{3}}{1} = 2-\sqrt{3}$.

93. $\tan\theta_2 = \tan\frac{\theta}{2} = \frac{3}{8}$; $\cos\theta = \frac{8}{x}$

$\tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$, so $\frac{3}{8} = \sqrt{\frac{1-\frac{8}{x}}{1+\frac{8}{x}}}$

$\frac{9}{64} = \frac{1-\frac{8}{x}}{1+\frac{8}{x}}$ $\qquad$ Square both members

$9\left(1+\frac{8}{x}\right) = 64\left(1-\frac{8}{x}\right)$ $\qquad$ If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$.

$9 + \frac{72}{x} = 64 - \frac{512}{x}$

$9x + 72 = 64x - 512$ $\qquad$ Multiply each member by x.

$584 = 55x$

$x = \frac{584}{55} = 10\frac{34}{55}$

97. $\cos 2\alpha - \cos 2\beta \quad = \quad -2\sin(\alpha+\beta)\cdot\sin(\alpha-\beta)$

$(2\cos^2\alpha - 1) - (2\cos^2\beta - 1)$ $\qquad$ $-2(\sin\alpha\cos\beta + \cos\alpha\sin\beta)(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$

$2\cos^2\alpha - 2\cos^2\beta$ $\qquad$ $-2(\sin^2\alpha\cos^2\beta - \cos^2\alpha\sin^2\beta)$

$-2[(1-\cos^2\alpha)(\cos^2\beta) - \cos^2\alpha(1-\cos^2\beta)]$

$-2[\cos^2\beta - \cos^2\alpha\cos^2\beta - \cos^2\alpha + \cos^2\alpha\cos^2\beta]$

$-2(\cos^2\beta - \cos^2\alpha)$

$2\cos^2\alpha - 2\cos^2\beta$

Exercise 7–4

1. $\tan\theta + 1 = 0$

$\tan\theta = -1$

$\theta' = \tan^{-1} 1 = \frac{\pi}{4}$ (45°)

tangent is negative in qII and qIV, so $\theta = \pi - \theta' = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ (180° − 45° = 135°)

or $2\pi - \theta' = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$ (360° − 45° = 315°).

5. $\sqrt{3}\tan\theta - 1 = 0$

$\tan\theta = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$;

$\theta' = \tan^{-1}\frac{\sqrt{3}}{3} = \frac{\pi}{6}$ (30°).

$\tan\theta > 0$ in qI and qIII, so $\theta = \theta' = \frac{\pi}{6}$ (30°) and

$\pi + \theta' = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$ (180° + 30° = 210°).

9. $3\sin^2\theta - 3 = 0$

$\sin^2\theta - 1 = 0$ $\qquad$ Divide each member by 3.

$\sin^2\theta = 1$

$\sin\theta = \pm 1$

$\sin\theta = 1$ at $\frac{\pi}{2}$ (90°) and $\sin\theta = -1$ at $\frac{3\pi}{2}$ (270°).

13. $(\cos\theta - 1)(\sin\theta + 1) = 0$

$\cos\theta - 1 = 0$ or $\sin\theta + 1 = 0$ $\qquad$ Zero product property.

$\cos\theta = 1$ or $\sin\theta = -1$

0 (0°) $\qquad$ $\frac{3\pi}{2}$ (270°)

17. $\sin^2\theta - \sin\theta = 0$

$\sin\theta(\sin\theta - 1) = 0$ $\qquad$ $a^2 - a = a(a-1)$

$\sin\theta = 0$ or $\sin\theta - 1 = 0$

0 (0°) and π (180°) $\qquad$ $\sin\theta = 1$

$\frac{\pi}{2}$ (90°)

21. $2\sin^2\theta + \sin\theta - 1 = 0$

$(2\sin\theta - 1)(\sin\theta + 1) = 0$ $\qquad$ Factor like $2u^2 + u - 1 = 0$.

$2\sin\theta - 1 = 0$ or $\sin\theta + 1 = 0$

$2\sin\theta = 1$ or $\sin\theta = -1$

$\sin\theta = \frac{1}{2}$ $\qquad$ $\frac{3\pi}{2}$ (270°)

$\theta' = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$; $\sin\theta > 0$ in qI and qII.

$\theta = \theta' = \frac{\pi}{6}$ (30°) and $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$ (150°)

25. $2\sin\theta\cos\theta - \sin\theta = 0$

$\sin\theta(2\cos\theta - 1) = 0$

$\sin\theta = 0$ or $2\cos\theta - 1 = 0$

0 (0°), π (180°) $\qquad$ $\cos\theta = \frac{1}{2}$

$\frac{\pi}{3}$ (60°) and $\frac{5\pi}{3}$ (300°)

(See problem 3.)

29. $\tan x\cot x = 0$

$\tan x = 0$ or $\cot x = 0$

$\frac{\sin x}{\cos x} = 0$ or $\frac{\cos x}{\sin x} = 0$

$\sin x = 0$ or $\cos x = 0$

0 (0°), π (180°) $\qquad$ $\frac{\pi}{2}$ (90°), $\frac{3\pi}{2}$ (270°)

33. $\tan x + \cot x = -2$

$\tan x + \frac{1}{\tan x} = -2$

$\tan^2 x + 1 = -2\tan x$ $\qquad$ Multiply each member by tan x; assume tan x ≠ 0.

$\tan^2 x + 2\tan x + 1 = 0$

$(\tan x + 1)^2 = 0$

$\tan x + 1 = 0$

$\tan x = -1$

$\frac{3\pi}{4}$ (135°) and $\frac{7\pi}{4}$ (315°) (Problem 1.)

37. $4\tan^2 x = 3\sec^2 x$

$4\tan^2 x = 3(\tan^2 x + 1)$

$4\tan^2 x = 3\tan^2 x + 3$

$\tan^2 x = 3$
$\tan x = \pm\sqrt{3}$
$\theta' = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$ (60°)

Since tan θ is both positive and negative there are solutions in each quadrant.

qI: $\theta = \theta' = \frac{\pi}{3}$ (60°)

qII: $\theta = \pi - \theta' = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ (180° − 60° = 120°)

qIII: $\theta = \pi + \theta' = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$ (180° + 60° = 240°)

qIV: $\theta = 2\pi - \theta' = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$ (360° − 60° = 300°)

41. $2\tan^2 x \sin x = \tan^2 x$
$2\tan^2 x \sin x - \tan^2 x = 0$
$\tan^2 x(2\sin x - 1) = 0$

$\tan^2 x = 0$	or	$2\sin x - 1 = 0$
$\tan x = 0$	or	$\sin x = \frac{1}{2}$
0 (0°), π (180°) (Prob. 19)		$\frac{\pi}{6}$ (30°) and $\frac{5\pi}{6}$ (150°) (Prob. 21.)

Problems 43 through 50 use:
If $ax^2 + bx + c = 0$, $c \neq 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

45. $\cot^2 x - 3\cot x - 2 = 0$
$a = 1, b = -3, c = -2$:
$\cot x = \dfrac{3 \pm \sqrt{9 - 4(-2)}}{2} = \dfrac{3 \pm \sqrt{17}}{2}$

$\cot x = \dfrac{3+\sqrt{17}}{2}$	$\cot x = \dfrac{3-\sqrt{17}}{2}$
$\tan x = \dfrac{2}{3+\sqrt{17}}$	$\tan x = \dfrac{2}{3-\sqrt{17}}$
$x' = \tan^{-1}\dfrac{2}{3+\sqrt{17}}$	$x' = \tan^{-1}\left\lvert\dfrac{2}{3+\sqrt{17}}\right\rvert$
$x' \approx 0.274$ (15.68°)	$x' = 1.059$ (60.68°)
tan x > 0 in qI and qIII.	tan x < 0 in qII and qIV.
$x = x' \approx 0.27$	$x = \pi - x' \approx \pi - 1.059 \approx 2.08$
$= \pi + x' \approx \pi + 0.274 \approx 3.42$	$= 2\pi - x' \approx 2\pi - 1.059 \approx 5.22$
$x° = x' \approx 15.7°$	$x° = 180° - x'$
$= 180° + x' \approx 180° + 15.7° \approx 195.7°$	$\approx 180° - 60.7° \approx 119.3°$
	$= 360° - x'$
	$\approx 360° - 60.7° \approx 299.3°$

49. $\tan x + 2\sec x = 3$
$\tan x = 3 - 2\sec x$
$(\tan x)^2 = (3 - 2\sec x)^2$ Since we are squaring both sides we must check all answers.
$\tan^2 x = 9 - 12\sec x + 4\sec^2 x$
$\sec^2 x - 1 = 9 - 12\sec x + 4\sec^2 x$
$3\sec^2 x - 12\sec x + 10 = 0$
$a = 3, b = -12, c = 10$: $\sec x = \dfrac{12 \pm \sqrt{144 - 120}}{6} = \dfrac{12 \pm 2\sqrt{6}}{6}$
$= \dfrac{6 \pm \sqrt{6}}{3}$

$\sec x = \dfrac{6+\sqrt{6}}{3}$	$\sec x = \dfrac{6-\sqrt{6}}{3}$
$\cos x = \dfrac{3}{6+\sqrt{6}} \approx 0.3551$	$\cos x = \dfrac{3}{6-\sqrt{6}} \approx 0.8449$
$x = \cos^{-1}\dfrac{3}{6+\sqrt{6}} \approx 1.208$ (69.2°)	$x = \cos^{-1}\dfrac{3}{6-\sqrt{6}} \approx 0.564$ (32.3°)
cos x > 0 in qI and qIV.	cos x > 0 in qI and qIV.
$x = x' \approx 1.21$	$x = x' \approx 0.56$
$= 2\pi - x' \approx 2\pi - 1.208 \approx 5.08$	$= 2\pi - x' \approx 2\pi - 0.56 \approx 5.72$
$x° = x' \approx 69.2°$	$x° = x' \approx 32.3°$
$= 360° - x' \approx 360° - 69.2° \approx 290.8°$	$= 360° - x'$
	$\approx 360° - 32.3° \approx 327.7$

As stated above we must check the solutions. 1.21 and 5.72 do not check:
tan(1.21) + 2 sec(1.21) ≈ 8.3, not 3, and
tan(5.72) + 2 sec(5.72) ≈ 1.7, not 3.
Thus the solutions are 5.08 (290.8°) and 0.56 (32.3°).

53. $\cot x = -\sqrt{3}$; $\tan x = -\frac{\sqrt{3}}{3}$

$x' = \tan^{-1}\frac{\sqrt{3}}{3} = \frac{\pi}{6}$ (30°).

Primary solutions are in qII and qIV: $\frac{5\pi}{6}$ (150°) and $\frac{11\pi}{6}$ (330°).
For all solutions we add $k\pi$ to these. Since the primary solutions differ by $k\pi$ ($k = 1$), we only need mention one primary solution to describe all solutions.
All solutions: $\frac{5\pi}{6} + k\pi$ (150° + k•180°).

57. $\tan x = 1$
$x = \tan^{-1} 1 = \frac{\pi}{4}$ (45°)

Primary solutions are in qI and qIII: $\frac{\pi}{4}$ (45°) and $\frac{5\pi}{4}$ (225°).
These differ by π (180°), so we can write all solutions with one of them: $\frac{\pi}{4} + k\pi$ (45° + k•180°).

61. $\sin\frac{x}{2} = \frac{\sqrt{3}}{2}$
$\left(\frac{x}{2}\right)' = \sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$ (60°)

Primary solutions to $\frac{x}{2}$ are in qI and qII. $\frac{\pi}{3}$ (60°) and $\frac{2\pi}{3}$ (120°).
All solutions:
$\frac{x}{2} = \frac{\pi}{3} + 2k\pi$ (60° + k•360°)
and $\frac{2\pi}{3} + 2k\pi$ (120° + k•360°).
$x = \frac{2\pi}{3} + 4k\pi$ (120° + k•720°)
and $\frac{4\pi}{3} + 4k\pi$ (240° + k•720°).

65. $3\cot 2x = \sqrt{3}$
$\cot 2x = \frac{\sqrt{3}}{3}$
$\tan 2x = \sqrt{3}$
$(2x)' = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$ (60°)
Primary solutions: $2x = \frac{\pi}{3}$ (60°) and $\frac{4\pi}{3}$ (240°)
All solutions:
$2x = \frac{\pi}{3} + k\pi$ (60° + k•180°) and $\frac{4\pi}{3} + k\pi$ (240° + k•180°)
$2x = \frac{\pi}{3} + k\pi$ (60° + k•180°)
The second expression is redundant.
$x = \frac{\pi}{6} + k\frac{\pi}{2}$ (30° + k•90°)

69. $2\cos 2x + 1 = 0$
$\cos 2x = -\frac{1}{2}$
$(2x)' = \cos^{-1}\frac{1}{2} = \frac{\pi}{3}$ (60°)
Primary solutions: $2x = \frac{2\pi}{3}$ (120°) and $\frac{4\pi}{3}$ (240°)
All solutions:
$2x = \frac{2\pi}{3} + 2k\pi$ (120° + k•360°)
and $\frac{4\pi}{3} + 2k\pi$ (240° + k•360)
$x = \frac{\pi}{3} + k\pi$ (60° + k•180°) and $\frac{2\pi}{3} + k\pi$ (120° + k•180)

73. $2\sin 2\theta = 1$
$\sin 2\theta = \frac{1}{2}$
$2\theta = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$ (30°)
Primary solutions:
$2\theta = \frac{\pi}{6} + 2k\pi$ (30° + k•360°)

and $\frac{5\pi}{6} + 2k\pi$ $(150° + k\cdot 360°)$

All solutions:

$\theta = \frac{\pi}{12} + k\pi$ $(15° + k\cdot 180°)$ and $\frac{5\pi}{12} + k\pi$ $(75° + k\cdot 180°)$

77. $\sqrt{3}\tan\frac{\theta}{4} = 1$

$\tan\frac{\theta}{4} = \frac{\sqrt{3}}{3}$

$\left(\frac{\theta}{4}\right)' = \tan^{-1}\frac{\sqrt{3}}{3} = \frac{\pi}{6}$ $(30°)$

Primary solutions: $\frac{\theta}{4} = \frac{\pi}{6}$ $(30°)$ and $\frac{7\pi}{6}$ $(210°)$

All solutions:

$\frac{\theta}{4} = \frac{\pi}{6} + k\pi$ $(30° + k\cdot 180°)$ and $\frac{7\pi}{6} + k\pi$ $(210° + k\cdot 180°)$

$\frac{\theta}{4} = \frac{\pi}{6} + k\pi$ $(30° + k\cdot 180°)$ The second expression is redundant.

$\theta = \frac{2\pi}{3} + 4k\pi$ $(120° + k\cdot 720°)$

81. $\sin 2\theta + \sin\theta = 0$

$2\sin\theta\cos\theta + \sin\theta = 0$

$\sin\theta(2\cos\theta + 1) = 0$

$\sin\theta = 0$	$2\cos\theta + 1 = 0$
0 $(0°)$, π $(180°)$	$\cos\theta = -\frac{1}{2}$
	$\frac{2\pi}{3}$ $(120°)$, $\frac{4\pi}{3}$ $(240°)$

85. $\sin\frac{\theta}{2} = \tan\frac{\theta}{2}$

$\sin\frac{\theta}{2} = \dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$

$\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \sin\frac{\theta}{2}$

$\sin\frac{\theta}{2}\cos\frac{\theta}{2} - \sin\frac{\theta}{2} = 0$

$\sin\frac{\theta}{2}(\cos\frac{\theta}{2} - 1) = 0$

$\sin\frac{\theta}{2} = 0$	$\cos\frac{\theta}{2} = 1$
$\frac{\theta}{2} = 0$ $(0°)$, π $(180°)$	$\frac{\theta}{2} = 0$ $(0°)$
$\theta = 0$ $(0°)$, 2π $(360°)$	$\theta = 0$ $(0°)$

Primary solutions: 0 $(0°)$

89. $\tan\frac{\theta}{2} = \cos\theta - 1$

$\pm\sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}} = \cos\theta - 1$

$\left(\pm\sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}}\right)^2 = (\cos\theta - 1)^2$ All solutions must be checked, since we square both members.

$\frac{1 - \cos\theta}{1 + \cos\theta} = \cos^2\theta - 2\cos\theta + 1$

$1 - \cos\theta = (\cos\theta + 1)(\cos^2\theta - 2\cos\theta + 1)$

$1 - \cos\theta = \cos^3\theta - \cos^2\theta - \cos\theta + 1$

$\cos^3\theta - \cos^2\theta = 0$

$\cos^2\theta(\cos\theta - 1) = 0$

$\cos^2\theta = 0$	$\cos\theta - 1 = 0$
$\cos\theta = 0$	$\cos\theta = 1$
$\frac{\pi}{2}$ $(90°)$, $\frac{3\pi}{2}$ $(270°)$	0 $(0°)$

These results must be checked because we squared both members in the first step.

$\frac{\pi}{2}$ does not check: $\tan\dfrac{\frac{\pi}{2}}{2} = \cos\frac{\pi}{2} - 1$

$\tan\frac{\pi}{4} = \cos\frac{\pi}{2} - 1$

$1 = 0 - 1$

$1 \neq -1$

The rest of the solutions check.

Thus the solutions are 0 $(0°)$ and $\frac{3\pi}{2}$ $(270°)$.

93. If $B = 0.7$, $x = 2$, $y = -8$, find A to the nearest 0.01.

$-8 = 2\cos A\cos 0.7 - 4\cos A\sin 0.7 - 8\sin A$

$-8 = (2\cos 0.7)\cos A - (4\sin 0.7)\cos A - 8\sin A$

$-8 = 1.5297\cos A - 2.5769\cos A - 8\sin A$

$-8 = -1.0472\cos A - 8\sin A$

$8\sin A - 8 = -1.0472\cos A$

$\sin A - 1 = -0.1309\cos A$ Divide each member by 8.

$\sin A = 1 - 0.1309\cos A$

$(\sin A)^2 = (1 - 0.1309\cos A)^2$

$\sin^2 A = 1 - 0.2618\cos A + 0.017134\cos^2 A$

$1 - \cos^2 A = 1 - 0.2618\cos A + 0.017134\cos^2 A$

$0 = 1.0171\cos^2 A - 0.2618\cos A$

$0 = \cos A\,(1.0171\cos A - 0.2618)$

$\cos A = 0$ or $1.0171\cos A - 0.2618 = 0$

$A = \cos^{-1}0$ or $1.0171\cos A = 0.2618$

$A = \frac{\pi}{2}$ or $\cos A = 0.25739$

$A \approx 1.310476103$

Thus A is $\frac{\pi}{2}$ or 1.31

Chapter 7 Review

1. $\frac{\cot\theta}{\cos\theta} = \csc\theta$

$\frac{\cos\theta}{\sin\theta}\cdot\frac{1}{\cos\theta}$

$\frac{1}{\sin\theta}$

$\csc\theta$

3. $\frac{\tan^2\theta}{\sec^2\theta - 1} = \sin^4\theta\csc^4\theta$

$\frac{\tan^2\theta}{\tan^2\theta}$ $\qquad \sin^4\theta\frac{1}{\sin^4\theta}$

1 $\qquad 1$

5. $\frac{\csc\theta\tan\theta}{\sin\theta} = \csc\theta\sec\theta$

$\csc\theta\tan\theta\frac{1}{\sin\theta}$

$\frac{1}{\sin\theta}\cdot\frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\sin\theta}$

$\frac{1}{\cos\theta}\cdot\frac{1}{\sin\theta}$

$\csc\theta\sec\theta$

7. $\csc\theta - \sec\theta$

$\frac{1}{\sin\theta} - \frac{1}{\cos\theta}$

$\frac{\cos\theta - \sin\theta}{\sin\theta\cos\theta}$

9. $\frac{\sec\theta}{\tan\theta - \cot\theta}$

$\dfrac{\frac{1}{\cos\theta}}{\frac{\sin\theta}{\cos\theta} - \frac{\cos\theta}{\sin\theta}}$

$\dfrac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta - \cos^2\theta}{\cos\theta\sin\theta}}$

$\frac{1}{\cos\theta}\cdot\frac{\cos\theta\sin\theta}{\sin^2\theta - \cos^2\theta}$

$\frac{\sin\theta}{\sin^2\theta - \cos^2\theta}$

11. $\frac{\sec^2\theta - 1}{\sec^2\theta}$

$\frac{\sec^2\theta}{\sec^2\theta} - \frac{1}{\sec^2\theta}$

$1 - \cos^2\theta$

$\sin^2\theta$

13. $\csc x - \tan x\cot x = \csc x - 1$

$\frac{1}{\sin x} - \tan x\frac{1}{\tan x}$

$\csc x - 1$

15. $\csc^2 x\sec^2 x(\cos^2 x - \sin^2 x)$

$\csc^2 x\sec^2 x\cos^2 x - \csc^2 x$

$\sec^2 x\sin^2 x$

$\csc^2 x\frac{1}{\cos^2 x}\cos^2 x - \frac{1}{\sin^2 x}$

$\sec^2 x\sin^2 x$

$\csc^2 x - \sec^2 x$

17. Left side:

$(\tan x - 1)(\csc^2 x - \cot^2 x)$

$(\tan x - 1)[(\cot^2 x + 1) - \cot^2 x]$

$(\tan x - 1)(1)$

$\tan x - 1$

Right side:

$\sec x(\sin x - \cos x)$

$\sec x\sin x - \sec x\cos x$

$\frac{1}{\cos x}\sin x - \frac{1}{\cos x}\cos x$

$\tan x - 1$

19. $\dfrac{1}{1+\csc x}+\dfrac{1}{1-\csc x}$

$\dfrac{(1-\csc x)+(1+\csc x)}{(1+\csc x)(1-\csc x)}$

$\dfrac{2}{1-\csc^2 x}$

$\dfrac{2}{-(\csc^2 x-1)}$

$-\dfrac{2}{\cot^2 x}$

$-2\tan^2 x$

21. $\tan^4 x+\tan^2 x$

$\tan^2 x(\tan^2 x+1)$

$\tan^2 x \cdot \sec^2 x$

$\dfrac{1}{\cot^2 x}\cdot \sec^2 x$

$\dfrac{\sec^2 x}{\cot^2 x}$

23. $\sin\frac{\pi}{12}=\sin\left(\frac{\pi}{4}-\frac{\pi}{6}\right)$

$=\sin\frac{\pi}{4}\cos\frac{\pi}{6}-\cos\frac{\pi}{4}\sin\frac{\pi}{6}$

$=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\cdot\frac{1}{2}$

$=\dfrac{\sqrt{6}-\sqrt{2}}{4}$

25. $\sin 75° = \sin(45°+30°)$

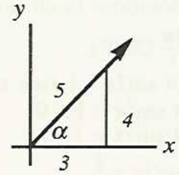
$=\sin 45°\cos 30° + \cos 45° \sin 30°$

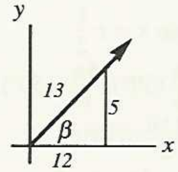
$=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\cdot\frac{1}{2}$

$=\dfrac{\sqrt{6}+\sqrt{2}}{4}$

27. $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

$=\frac{4}{5}\cdot\frac{12}{13}+\frac{3}{5}\cdot\frac{5}{13}=\frac{63}{65}.$





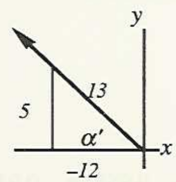
29. $\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

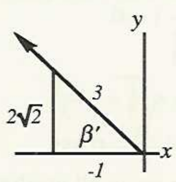
$=\dfrac{-\frac{5}{12}-(-2\sqrt{2})}{1+\left(-\frac{5}{12}\right)(-2\sqrt{2})}=\dfrac{-\frac{5}{12}+2\sqrt{2}}{1+\frac{5\sqrt{2}}{6}}\cdot\dfrac{12}{12}=\dfrac{-5+24\sqrt{2}}{12+10\sqrt{2}}$

$=\frac{1}{2}\cdot\dfrac{-5+24\sqrt{2}}{6+5\sqrt{2}}\cdot\dfrac{6-5\sqrt{2}}{6-5\sqrt{2}}$

$=\frac{1}{2}\cdot\dfrac{-30+25\sqrt{2}+144\sqrt{2}-120(2)}{36-25(2)}$

$=\dfrac{-270+169\sqrt{2}}{-28}=\dfrac{270-169\sqrt{2}}{28}$





31. $\sin\left(\frac{\pi}{4}-\theta\right)=\sin\frac{\pi}{4}\cos\theta-\cos\frac{\pi}{4}\sin\theta$

$=\frac{\sqrt{2}}{2}\cos\theta-\frac{\sqrt{2}}{2}\sin\theta=\frac{\sqrt{2}}{2}(\cos\theta-\sin\theta)$

33. $\sin(\theta+2\pi) = \sin\theta\cos 2\pi+\cos\theta\sin 2\pi$

$=\sin\theta\,(1)+\cos\theta\,(0)=\sin\theta$

$x=\sqrt{10^2-8^2}=6.\ \ r=\sqrt{6^2+x^2}=\sqrt{6^2+6^2}$

$=6\sqrt{2}.$

$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

$=\frac{8}{10}\cdot\frac{x}{r}-\frac{x}{10}\cdot\frac{6}{r}=\frac{4}{5}\cdot\frac{6}{6\sqrt{2}}-\frac{3}{5}\cdot\frac{6}{6\sqrt{2}}$

$=\frac{6}{30\sqrt{2}}=\frac{1}{5\sqrt{2}}=\frac{\sqrt{2}}{10}.$

35. $\cos^2\alpha-\sin^2\alpha=\cos 2\alpha$

$\cos^2 62°-\sin^2 62°=\cos\theta$

If α is $62°$, then $2\alpha=124°=\theta$

37. $\tan\theta=\dfrac{2\tan\frac{7\pi}{12}}{1-\tan^2\frac{7\pi}{12}}$

$\tan 2\alpha=\dfrac{2\tan\alpha}{1-\tan^2\alpha}$; $\alpha=\frac{7\pi}{12}$, $2\alpha=\frac{7\pi}{6}=\theta$

39. $a\sin b\theta=6\sin\frac{\theta}{2}\cos\frac{\theta}{2}$; find a and b.

$\sin 2\alpha=2\sin\alpha\cos\alpha$

Transform both members so the right member is $6\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$3\sin 2\alpha=6\sin\alpha\cos\alpha$ Multiply each member by 3

$3\sin 2\left(\frac{\theta}{2}\right)=6\sin\frac{\theta}{2}\cos\frac{\theta}{2}$ Replace α by $\frac{\theta}{2}$.

$3\sin\theta=6\sin\frac{\theta}{2}\cos\frac{\theta}{2}.$

Since $3\sin\theta$ corresponds to $a\sin b\theta$, $a=3$ and $b=1$.

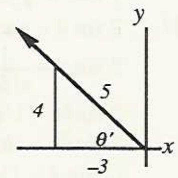
41. Given $\sin\theta=\frac{4}{5}$, θ lies in quadrant II. Find the exact value of

(a) $\cos 2\theta$; (b) $\tan 2\theta$.

(a) $\cos 2\theta=2\cos^2\theta-1=2\left(-\frac{3}{5}\right)^2-1=\frac{18}{25}-1=-\frac{7}{25}$

(b) $\tan 2\theta=\dfrac{2\tan\theta}{1-\tan^2\theta}=\dfrac{2\left(-\frac{4}{3}\right)}{1-\left(-\frac{4}{3}\right)^2}$

$=\dfrac{-\frac{8}{3}}{1-\frac{16}{9}}\cdot\frac{9}{9}=\frac{-24}{9-16}=\frac{24}{7}.$



43. $1+\cos 2x=2\cos^2 x$

$1+(2\cos^2 x-1)$

$2\cos^2 x$

45. $\cos\frac{\pi}{8}=\cos\dfrac{\frac{\pi}{4}}{2}=\pm\sqrt{\dfrac{1+\cos\frac{\pi}{4}}{2}}=\sqrt{\dfrac{1+\frac{\sqrt{2}}{2}}{2}}=\sqrt{\dfrac{2+\sqrt{2}}{4}}$

$=\dfrac{\sqrt{2+\sqrt{2}}}{2}.$

47. Given $\sin x=-\frac{2}{3}$, $3\pi<x<\frac{7\pi}{2}$. Find (a) $\cos\frac{x}{2}$; (b) $\tan\frac{x}{2}$.

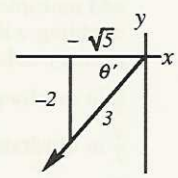
$3\pi<x<\frac{7\pi}{2}$, so x terminates in quadrant III; also, $\frac{3\pi}{2}<\frac{x}{2}<\frac{7\pi}{4}$, so $\frac{x}{2}$ terminates in quadrant IV, where cosine is positive and tangent is negative.

$\cos\frac{x}{2}=\sqrt{\dfrac{1+\cos x}{2}}=\sqrt{\dfrac{1+(-\frac{\sqrt{5}}{3})}{2}}=\sqrt{\dfrac{3-\sqrt{5}}{6}}$

$=\dfrac{\sqrt{3-\sqrt{5}}}{\sqrt{6}}\cdot\dfrac{\sqrt{6}}{\sqrt{6}}=\dfrac{\sqrt{18-6\sqrt{5}}}{6}$

$\tan\frac{x}{2}=\dfrac{1-\cos\theta}{\sin\theta}=\dfrac{1-(\frac{-\sqrt{5}}{3})}{-\frac{2}{3}}\cdot\frac{3}{3}$

$=\dfrac{3+\sqrt{5}}{-2}=-\dfrac{3+\sqrt{5}}{2}$



49. $\sec^2\frac{\theta}{2}-\tan^2\frac{\theta}{2}$

$(\tan^2\frac{\theta}{2}+1)-\tan^2\frac{\theta}{2}$

1

Recall that $\sec^2\alpha=\tan^2\alpha+1$.

51. $3\cot^2 x-1=0$

$\cot^2 x=\frac{1}{3}$

$\cot x=\pm\dfrac{1}{\sqrt{3}}$

$\tan x=\pm\sqrt{3}$

$x'=\tan^{-1}\sqrt{3}=\frac{\pi}{3}\ (60°)$

Solutions in all four quadrants: $\frac{\pi}{3}$ (60°), $\frac{2\pi}{3}$ (120°), $\frac{4\pi}{3}$ (240°), $\frac{5\pi}{3}$ (300°).

53. $(4\sin^2 x - 1)(\sec x - 2) = 0$

$4\sin^2 x - 1 = 0$	$\sec x - 2 = 0$
$4\sin 2x = 1$	$\sec x = 2$
$\sin 2x = \frac{1}{4}$	$\cos x = \frac{1}{2}$
$\sin x = \pm\frac{1}{2}$	
$\frac{\pi}{6}$ (30°), $\frac{5\pi}{6}$ (150°), $\frac{7\pi}{6}$ (210°), $\frac{11\pi}{6}$ (330°)	$\frac{\pi}{3}$ (60°), $\frac{5\pi}{3}$ (300°)

55. $\sec^2\theta - 4 = 0$
$\sec^2\theta = 4$
$\sec\theta = \pm 2$
$\cos\theta = \pm\frac{1}{2}$
$\frac{\pi}{3}$ (60°), $\frac{2\pi}{3}$ (120°), $\frac{4\pi}{3}$ (240°), $\frac{5\pi}{3}$, (300°)

57. $2\sin\theta - \csc\theta + 1 = 0$
$2\sin\theta - \frac{1}{\sin\theta} + 1 = 0$
$2\sin^2\theta - 1 + \sin\theta = 0$
$2\sin^2\theta + \sin\theta - 1 = 0$
$(2\sin\theta - 1)(\sin\theta + 1) = 0$

$\sin\theta = \frac{1}{2}$	$\sin\theta = -1$
$\frac{\pi}{6}$ (30°), $\frac{5\pi}{6}$ (150°)	$\frac{3\pi}{2}$ (270°)

59. $2\sin^2\theta - 3\cos\theta = 3$
$2(1 - \cos^2\theta) - 3\cos\theta = 3$
$2 - 2\cos^2\theta - 3\cos\theta = 3$
$2\cos^2\theta + 3\ \text{co s}\theta + 1 = 0$
$(2\cos\theta + 1)(\cos\theta + 1) = 0$

$2\cos\theta + 1 = 0$	$\cos\theta + 1 = 0$
$\cos\theta = -\frac{1}{2}$	$\cos\theta = -1$
$\frac{2\pi}{3}$ (120°), $\frac{4\pi}{3}$ (240°)	π (180°)

61. $2\cos\frac{x}{2} - \sqrt{3} = 0$
$\cos\frac{x}{2} = \frac{\sqrt{3}}{2}$
$\frac{x}{2} = \cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}$ (30°)
Since we will be doubling final solutions, there is no need to add multiples of 2π to primary solutions. (Otherwise, doubling will produce values greater than 4π, which are not primary solutions.)
$\frac{x}{2}$ in quadrant I: $\frac{x}{2} = \frac{\pi}{6}$ (30°), so $x = \frac{\pi}{3}$ (60°);
$\frac{x}{2}$ in quadrant IV: $\frac{x}{2} = \frac{11\pi}{6}$ (330°), so $x = \frac{11\pi}{3}$ (660°).
Discarding the solution greater than 2π, we obtain $x = \frac{\pi}{3}$ (60°).

63. $\sin^2\frac{x}{4} = \frac{1}{2}$
$\sin\frac{x}{4} = \pm\frac{\sqrt{2}}{2}$
$\left(\frac{x}{4}\right)' = \sin^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$ (45°)
$\frac{x}{4} = \frac{\pi}{4}$ (45°), $\frac{3\pi}{4}$ (135°), $\frac{5\pi}{4}$ (225°), $\frac{7\pi}{4}$ (315°), so, multiplying by 4 and discarding non-primary solutions,
$x = \pi$ (180°).

65. $2\cos 5\theta = 1$
$\cos 5\theta = \frac{1}{2}$
$(5\theta)' = \cos^{-1}\frac{1}{2} = \frac{\pi}{3}$ (60°).
Since we will divide solutions by 5, we should find solutions to 5θ out to 8π ($10\pi \div 5 = 2\pi$).
5θ in quadrant I:
$5\theta = \frac{\pi}{3}, \frac{\pi}{3} + 2\pi, \frac{\pi}{3} + 4\pi, \frac{\pi}{3} + 6\pi, \frac{\pi}{3} + 8\pi$ or 60°, 60° + 360°, 60° + 720°, 60° + 1080°, 60° + 1440°
5θ in quadrant IV:
$5\theta = \frac{5\pi}{3}, \frac{5\pi}{3} + 2\pi, \frac{5\pi}{3} + 4\pi, \frac{5\pi}{3} + 6\pi, \frac{5\pi}{3} + 8\pi$ or 300°, 300° + 360°, 300° + 720°, 300° + 1080°, 300° + 1440°
Dividing solutions by 5 we obtain:
$\frac{\pi}{15}, \frac{7\pi}{15}, \frac{13\pi}{15}, \frac{19\pi}{15}, \frac{5\pi}{3}, \frac{\pi}{3}, \frac{11\pi}{15}, \frac{17\pi}{15}, \frac{23\pi}{15}, \frac{29\pi}{15}$ or 12°, 84°, 156°, 228°, 300°, 60°, 132°, 204°, 276°, 348°.

67. $3\cos^2 x - 1 = 0$
$\cos^2 x = \frac{1}{3}$
$\cos x = \pm\sqrt{\frac{1}{3}} = \pm\frac{\sqrt{3}}{3}$
$x' = \cos^{-1}\frac{\sqrt{3}}{3} \approx 0.955$
Solutions are in all quadrants: $0.955, \pi - 0.955, \pi + 0.955, 2\pi - 0.955$, so $x \approx 0.96, 2.19, 4.10, 5.33$

69. $\cos\frac{x}{2} - \sin x = 0$
$\left(\pm\sqrt{\frac{1 + \cos x}{2}}\right) - \sin x = 0$
$\left(\pm\sqrt{\frac{1 + \cos x}{2}}\right)^2 = (\sin x)^2$ All solutions must be checked since we are squaring both members.
$\frac{1 + \cos x}{2} = \sin^2 x$
$\frac{1 + \cos x}{2} = 1 - \cos^2 x$
$1 + \cos x = 2 - 2\cos^2 x$
$2\cos^2 x + \cos x - 1 = 0$
$(2\cos x - 1)(\cos x + 1) = 0$

$2\cos x - 1 = 0$	$\cos x + 1 = 0$
$\cos x = \frac{1}{2}$	$\cos x = -1$
$x = \frac{\pi}{3}, \frac{5\pi}{3}$	$x = \pi$

All three solutions check.

71. $3\sin^2 2x - \sin 2x - 2 = 0$
$(3\sin 2x + 2)(\sin 2x - 1) = 0$

$3\sin 2x + 2 = 0$	$\sin 2x - 1 = 0$
$\sin 2x = -\frac{2}{3}$	$\sin 2x = 1$
$(2x)' = \sin^{-1}\frac{2}{3} \approx 0.7297$	$(2x)' = \frac{\pi}{2}$
Primary solutions for $2x$: are in qIII, qIV:	Primary solutions for $2x$:
$2x = \pi + (2x)', 2\pi - (2x)'$	$2x = \frac{\pi}{2}, 2\pi + \frac{\pi}{2}$
$2x \approx 3.8713, 5.5535$	
Solutions to $2x$ out to 4π:	

$2x \approx 3.8713, 5.5535, 3.8713 + 2\pi, 5.5535 + 2\pi$

$2x \approx 3.8713, 5.5535, 10.1545, 11.8366$

$x \approx 1.94, 2.78, 5.08, 5.92$

$2x = \frac{\pi}{2}, \frac{5\pi}{2}$

$x = \frac{\pi}{4}, \frac{5\pi}{4}$

Thus the primary solutions are the approximate values 1.94, 2.78, 5.08, 5.92 and the exact values $\frac{\pi}{4}, \frac{5\pi}{4}$.

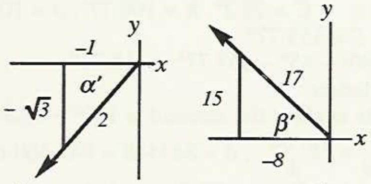
Note: If the coefficients of the original equation were measured quantities we could not claim that the last four solutions were exact.

Chapter 7 Test

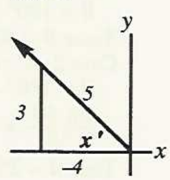
1. $\csc^2 x \sin x \cos x$
$\frac{1}{\cos^2 x} \sin x \cos x$
$\frac{1}{\cos x} \sin x$
$\frac{\sin x}{\cos x}$
$\cot x$

3. $\cot x - 2 \tan x = 1$
$\cot \frac{\pi}{4} - 2 \tan \frac{\pi}{4} = 1$
$1 - 2(1) = 1$
$-1 \neq 1$

5. $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$
$= -\frac{\sqrt{3}}{2}\left(-\frac{8}{17}\right) + \left(-\frac{1}{2}\right)\frac{15}{17} = \frac{8\sqrt{3} - 15}{34}$



7. $3\pi < x < \frac{5\pi}{2}$, so x is in quadrant II, and $\frac{3\pi}{2} < \frac{x}{2} < \frac{5\pi}{4}$, so $\frac{x}{2}$ is in quadrant III, so $\tan \frac{x}{2} > 0$. $\tan \frac{x}{2} = \sqrt{\frac{1 - \cos x}{1 + \cos x}}$
$= \sqrt{\frac{1 - \left(-\frac{4}{5}\right)}{1 + \left(-\frac{4}{5}\right)}} = \sqrt{\frac{\frac{9}{5}}{\frac{1}{5}}} = 3.$



9. $\frac{1 + \cot \theta}{\csc \theta}$
$\frac{1}{\csc \theta} + \frac{\cot \theta}{\csc \theta}$
$\sin \theta + \cot \theta \frac{1}{\csc \theta}$
$\sin \theta + \frac{\cos \theta}{\sin \theta} \cdot \sin \theta$
$\sin \theta + \cos \theta$

11. $\tan^4 x + \tan^2 x$
$\tan^2\theta(\tan^2\theta + 1)$
$\tan^2\theta \sec^2\theta$
$\frac{1}{\cot^2\theta} \sec^2\theta$
$\frac{\sec^2\theta}{\cot^2\theta}$

13. $\cos 2x - \sin 2x$
$2 \cos^2 x - 1 - (2 \sin x \cos x)$
$2 \cos^2 x - 2 \sin x \cos x - 1$
$2\cos x(2 \cos x - \sin x) - 1$

15. $(\cot \theta - \sqrt{3})(\sec \theta + 2) = 0$
$\cot \theta - \sqrt{3} = 0$
$\cot \theta = \sqrt{3}$
$\tan \theta = \frac{\sqrt{3}}{3}$
$\theta' = \tan^{-1} \frac{\sqrt{3}}{3} = \frac{\pi}{6} (30°)$
$\theta = \frac{\pi}{6} (30°), \frac{7\pi}{6} (210°)$

$\sec \theta + 2 = 0$
$\sec \theta = -2$
$\cos \theta = -\frac{1}{2}$
$\theta' = \cos^{-1} \frac{1}{2} = \frac{\pi}{3} (60°)$
$\frac{2\pi}{3} (120°), \frac{4\pi}{3} (240°)$

17. $\sin^2 3x = \frac{1}{2}$
$\sin 3x = \pm \frac{\sqrt{2}}{2}$
$(3x)' = \sin^{-1} \frac{\sqrt{2}}{2} = \frac{\pi}{4}$
$3x = \frac{\pi}{4}, \frac{\pi}{4} + 2\pi, \frac{\pi}{4} + 4\pi, \frac{3\pi}{4}, \frac{3\pi}{4} + 2\pi, \frac{3\pi}{4} + 4\pi, \frac{5\pi}{4}, \frac{5\pi}{4} + 2\pi, \frac{5\pi}{4} + 4\pi, \frac{7\pi}{4}, \frac{7\pi}{4} + 2\pi, \frac{7\pi}{4} + 4\pi$
$x = \frac{\pi}{12}, \frac{\pi}{12} + \frac{2\pi}{3}, \frac{\pi}{12} + \frac{4\pi}{3}, \frac{3\pi}{12}, \frac{3\pi}{12} + \frac{2\pi}{3}, \frac{3\pi}{12} + \frac{4\pi}{3}, \frac{5\pi}{12}, \frac{5\pi}{12} + \frac{2\pi}{3}, \frac{5\pi}{12} + \frac{4\pi}{3}, \frac{7\pi}{12}, \frac{7\pi}{12} + \frac{2\pi}{3}, \frac{7\pi}{12} + \frac{4\pi}{3}$
$= \frac{\pi}{12}, \frac{3\pi}{4}, \frac{17\pi}{12}, \frac{\pi}{4}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{21\pi}{12}, \frac{7\pi}{12}, \frac{5\pi}{4}, \frac{23\pi}{12}$

Chapter 8

Exercise 8–1

1. $a = 12.5, A = 35°, B = 49°$
$C = 180° - A - B = 96°$
$$\frac{\sin 35°}{12.5} = \frac{\sin 49°}{b} = \frac{\sin 96°}{c}$$
$\frac{\sin 35°}{12.5} = \frac{\sin 49°}{b}$ $\qquad$ $\frac{\sin 35°}{12.5} = \frac{\sin 96°}{c}$
$b = \frac{12.5 \sin 49°}{\sin 35°}$ $\qquad$ $c = \frac{12.5 \sin 96°}{\sin 35°}$
$b \approx 16.4$ $\qquad$ $c \approx 21.7$

5. $b = 92.5, A = 47°, B = 100°$
$C = 180° - 47° - 100° = 33°$
$$\frac{\sin 47°}{a} = \frac{\sin 100°}{92.5} = \frac{\sin 33°}{c}$$
$\frac{\sin 100°}{92.5} = \frac{\sin 33°}{c}$ $\qquad$ $\frac{\sin 47°}{a} = \frac{\sin 100°}{92.5}$
$c = \frac{92.5 \sin 33°}{\sin 100°}$ $\qquad$ $a = \frac{92.5 \sin 47°}{\sin 100°}$
$c \approx 51.2$ $\qquad$ $a \approx 68.7$

9. $c = 5.00, A = 100°, B = 45°$
$C = 180° - 100° - 45° = 35°$
$$\frac{\sin 100°}{a} = \frac{\sin 45°}{b} = \frac{\sin 35°}{5}$$
$\frac{\sin 100°}{a} = \frac{\sin 35°}{5}$ $\qquad$ $\frac{\sin 45°}{b} = \frac{\sin 35°}{5}$
$a = \frac{5 \sin 100°}{\sin 35°}$ $\qquad$ $b = \frac{5 \sin 45°}{\sin 35°}$
$a \approx 8.58$ $\qquad$ $b \approx 6.16$

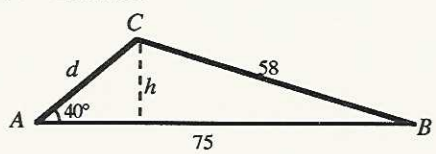
13. $a = 4.25, c = 2.86, A = 132°$
$$\frac{\sin 132°}{4.25} = \frac{\sin B}{b} = \frac{\sin C}{2.86}$$
$\frac{\sin 132°}{4.25} = \frac{\sin C}{2.86}$ so $\sin C = \frac{2.86 \sin 132°}{4.25}$,
$C' = \sin^{-1} \frac{2.86 \sin 132°}{4.25} \approx 30.01°$
$C \approx 30.01°$ or $180° - 30.01° \approx 149.99°$
Case 1: $C \approx 30.01°$
$B \approx 180° - 132° - 30.01° \approx 17.99°$
$\frac{\sin 132°}{4.25} \approx \frac{\sin 17.99°}{b}$
$b \approx 1.77$
Thus, the solution is $B \approx 18.0°, C \approx 30.0°, b \approx 1.77$.
Case 2: $C \approx 149.99°$
$B = 180° - 132° - 149.99° \approx -101.99$ (No solution.)

17. $a = 4, b = 22, A = 30°$
$$\frac{\sin 30°}{4} = \frac{\sin B}{22} = \frac{\sin C}{c}$$
$\frac{\sin 30°}{4} = \frac{\sin B}{22}$ so $\sin B = \frac{22 \sin 30°}{4}$,
$B' = \sin^{-1} \frac{22 \sin 30°}{4} = \sin^{-1} 2.75$
B' is undefined, since $0 < \sin B < 1$, and we have $\sin B = 2.75$ from this data.
Thus, there is no solution possible.

21. $c = 5.00, b = 8.00, B = 45.0°$
$$\frac{\sin A}{a} = \frac{\sin 45°}{8} = \frac{\sin C}{5}$$
$\frac{\sin 45°}{8} = \frac{\sin C}{5}$, so $\sin C = \frac{5 \sin 45°}{8}$
$C' = \sin^{-1} \frac{5 \sin 45°}{8} \approx 26.23°$.
$C \approx 26.23°$ or $180° - 26.23° \approx 153.77°$
Case 1: $C \approx 26.23°$
$A \approx 180° - 45° - 26.23° \approx 108.77°$
$\frac{\sin 108.77°}{a} = \frac{\sin 45°}{8}$
$a \approx 10.71$
Solution: $C \approx 26.2°, A \approx 108.77°, a \approx 10.71$
Case 2: $C \approx 153.77°$
$A \approx 180° - 45° - 153.77° \approx -18.77°$
No solution.

25. The acute angle at the asteroid is $180° - 81.5° - 88° = 10.5°$.
$\frac{\sin 10.5°}{153\ 800} = \frac{\sin 88°}{d}$; $d \approx 843449 \approx 843{,}400$ miles.

29. $\frac{\sin 40°}{58} = \frac{\sin C}{75}$; $\sin C = \frac{75 \sin 40°}{58}$; $C' \approx 56.2°$.
$C \approx 56.2°$ or $180° - 56.2° \approx 123.8°$.
Case 1: $C \approx 56.2°$
$B = 180° - 40° - 56.2° \approx 83.8°$
Since $B < 40°$, we solve the second case.
Case 2: $C \approx 123.8°$
$B = 180° - 40° - 123.8° \approx 16.2°$
$\frac{\sin 40°}{58} = \frac{\sin 16.2°}{d}$; $d \approx 25.17$
Thus $d \approx 25$ miles.



33. Let (x, y) be the point at B. It is on the terminal side of angle A. Then $\cos A = \frac{x}{r}$, where r is the length of AB. But then $r = c$, so $\cos A = \frac{x}{c}$.
Using right triangles we see that in each figure $\cos C = \frac{b-x}{a}$. Note that when A is obtuse (the right hand figure) x is negative, so $b - x$ is the length of $|b| + |x|$.
$\cos C = \frac{b-x}{a}$ $\qquad$ $\cos A = \frac{x}{c}$
$a \cos C = b - x$ $\qquad$ $c \cos A = x$
$b - a \cos C = x$
$b - a \cos C = c \cos A$
$c \cos A = b - a \cos C$
$b = c \cos A + a \cos C$
Thus (2) is true.

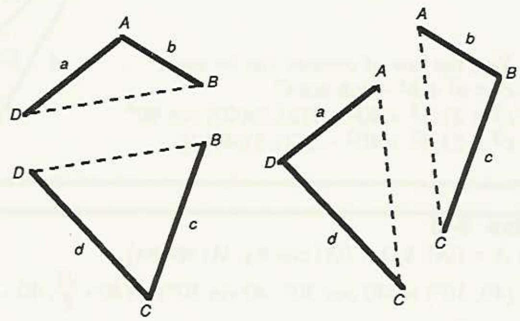
(1) and (3) can be shown true by putting angles B and C in standard position and proceeding in the same manner. In fact this is not really necessary, since the labeling in a triangle is arbitrary, and thus, for example, we could obtain (1) by changing the label B to A, C to B, and A to C, and labeling the sides appropriately.

37. (a) It can be seen that the sum of the area of the four triangles shown in the figure is
$\frac{1}{2} ab \sin A + \frac{1}{2} cd \sin C + \frac{1}{2} ad \sin D + \frac{1}{2} bc \sin B$.
This total is twice as large as the total area of the four–sided figure, so the area of the four–sided figure is
$\frac{1}{2}\left(\frac{1}{2} ab \sin A + \frac{1}{2} cd \sin C + \frac{1}{2} ad \sin D + \frac{1}{2} bc \sin B\right)$
or $\frac{1}{4}(ab \sin A + ad \sin D + bc \sin B + cd \sin C)$.

(b) The difference between the Egyptian formula $\frac{1}{4}(ab + ad + bc + cd)$ and the correct formula $\frac{1}{4}(ab \sin A + ad \sin D + bc \sin B + cd \sin C)$ is the factors $\sin A$, $\sin B$, $\sin C$, and $\sin D$. Since we assumed each angle is between 0° and 180° the value of the sine of each angle is between 0 and 1.

Thus,
$$ab \geq ab \sin A$$
$$ad \geq ad \sin D$$
$$bc \geq bc \sin B$$
$$cd \geq cd \sin C$$
$$ab + ad + bc + cd \geq ab \sin A + ad \sin D + bc \sin B + cd \sin C,$$
$$\tfrac{1}{4}(ab + ad + bc + cd) \geq \tfrac{1}{2}(ab \sin A + ad \sin D + bc \sin B + cd \sin C).$$

If the figure is a rectangle $A = B = C = D = 90°$, and $\sin A = \sin B = \sin C = \sin D = 1$, so both expressions give the same value.



Exercise 8–2

1. $a = 3.2, b = 5.9, C = 39.4°$

$c^2 = a^2 + b^2 - 2ab \cos C$

$c^2 = 3.2^2 + 5.9^2 - 2(3.2)(5.9) \cos 39.4°$

$c = \sqrt{3.2^2 + 5.9^2 - 2(3.2)(5.9) \cos 39.4°} \approx 3.9839$

3.2 $\boxed{x^2}$ $\boxed{+}$ 5.9 $\boxed{x^2}$ $\boxed{-}$ 2 $\boxed{\times}$ 3.2 $\boxed{\times}$ 5.9 $\boxed{\times}$ 39.4 $\boxed{\cos}$ $\boxed{=}$ $\boxed{\sqrt{x}}$

$\boxed{\text{TI-81}}$ $\boxed{\sqrt{\ }}$ $\boxed{(}$ 3.2 $\boxed{x^2}$ $\boxed{+}$ 5.9 $\boxed{x^2}$ $\boxed{-}$ 2 $\boxed{\times}$ 3.2 $\boxed{\times}$ 5.9 $\boxed{\times}$ $\boxed{\text{COS}}$ 39.4 $\boxed{)}$ $\boxed{\text{ENTER}}$

$\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

$\dfrac{\sin A}{3.2} = \dfrac{\sin B}{5.9} = \dfrac{\sin 39.4°}{3.9839}$

$\sin A = \dfrac{3.2 \sin 39.4°}{3.9839}$; $A \approx 30.653°$; A is acute since it is the smallest angle in the triangle.

3.2 $\boxed{\times}$ 39.4 $\boxed{\sin}$ $\boxed{\div}$ 3.9839 $\boxed{=}$ $\boxed{\text{SHIFT}}$ $\boxed{\sin}$

$B = 180° - A - C \approx 180° - 39.4° - 30.7° \approx 109.9°$.

Thus, $c \approx 4.0, A \approx 30.7°, B \approx 109.9°$.

5. $a = 31.4, c = 17.0, B = 100.3°$

$b^2 = a^2 + c^2 - 2ac \cos B$

$b^2 = 31.4^2 + 17^2 - 2(31.4)(17) \cos 100.3°$

$b = \sqrt{31.4^2 + 17^2 - 2(31.4)(17) \cos 100.3°} \approx 38.286$

$\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

$\dfrac{\sin A}{31.4} = \dfrac{\sin 100.3}{38.286} = \dfrac{\sin C}{17}$; Since B is obtuse, A and C are acute. We can find either next.

$\sin A \approx \dfrac{31.4 \sin 100.3°}{38.286}$; $A \approx 53.8°$

$C \approx 180° - 53.8° - 100.3° \approx 25.9°$

Thus $b \approx 38.3, A \approx 53.8°, C \approx 25.9°$.

9. $a = 0.214, b = 0.500, c = 0.399$

$b^2 = a^2 + c^2 - 2ac \cos B$, so $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

$= \dfrac{0.214^2 + 0.399^2 - 0.5^2}{2(0.214)(0.399)}$,

so $B \approx 105.28° \approx 105.3°$.

$\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

$\dfrac{\sin A}{0.214} = \dfrac{\sin 105.28°}{0.5} = \dfrac{\sin C}{0.399}$; Since B is the largest angle, A and C are acute.

$\sin A = \dfrac{\sin 105.28°}{0.5}(0.214)$, so $A \approx 24.4°$

$C \approx 180° - 24.4° - 105.3° \approx 50.3°$.

13. $b = 61.3, c = 43.9, A = 24.5°$

$a^2 = b^2 + c^2 - 2bc \cos A$

$a^2 = 61.3^2 + 43.9^2 - 2(61.3)43.9 \cos 24.5°$

$a \approx 28.06 \approx 28.1$

$\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

$\dfrac{\sin 24.5°}{28.06} = \dfrac{\sin B}{61.3} = \dfrac{\sin C}{43.9}$; B may be obtuse or acute. Therefore C must be acute.

$\sin C \approx \dfrac{\sin 24.5°}{28.06}(43.9)$, so $C \approx 40.5°$

$B \approx 180° - 40.5° - 24.5° \approx 115.0°$.

17. $a = 235, b = 194, c = 354$

$c^2 = a^2 + b^2 - 2ab \cos C$, so $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

$= \dfrac{235^2 + 194^2 - 354^2}{2(235)(194)}$,

$C \approx 110.85° \approx 110.9°$

$\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

$\dfrac{\sin A}{235} = \dfrac{\sin B}{194} = \dfrac{\sin 110.85°}{354}$; A and B are acute.

$\sin A \approx \dfrac{\sin 110.85°}{354}(235)$; $A \approx 38.3°$

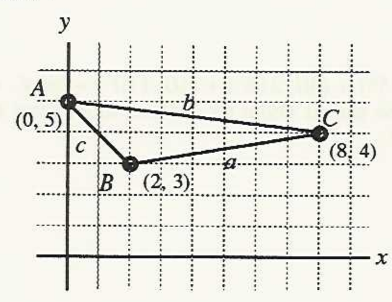
$B \approx 180° - 38.3° - 110.9° \approx 30.8°$.

21. If d is the distance, then $d^2 = 421^2 + 372^2 - 2(421)(372) \cos 48.2°$, $d \approx 326.9$ ft.

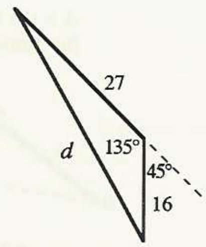
25. Use the distance formula, which states that for two points (x_1, y_1), (x_2, y_2), the distance, d, between them is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. We find that $a = \sqrt{37}$, $b = \sqrt{65}$, $c = \sqrt{8}$. The largest angle is opposite the longest side, b. Thus $b^2 = a^2 + c^2 - 2ac \cos B$, so

$\cos B = \dfrac{a^2 + c^2 - b^2}{2ac} = \cos B = \dfrac{37 + 8 - 65}{2(\sqrt{37})(\sqrt{8})}, \approx -0.5812$,

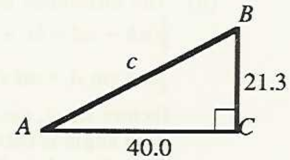
$B \approx 125.5°$.



29. $d^2 = 27^2 + 16^2 - 2(27)(16)\cos 135°$; $d \approx 39.9$ miles.



33. Yes, the law of cosines can be used:
$c^2 = a^2 + b^2 - 2ab\cos C$
$c^2 = 21.3^2 + 40^2 - 2(21.3)(40)\cos 90°$
$c^2 = 21.3^2 + 40^2 - 2(21.3)(40)(0)$
$c^2 = 21.3^2 + 40^2$
$c \approx 45.3$
Since $\cos 90° = 0$, the Law of Cosines is the same as the Pythagorean Theorem when the angle used is 90°.



Exercise 8–3

We use $A = (|A|, \theta_A) = (|A|\cos\theta_A, |A|\sin\theta_A)$.

1. $(40, 30°) = (40\cos 30°, 40\sin 30°) = (40\cdot\frac{\sqrt{3}}{2}, 40\cdot\frac{1}{2}) = (20\sqrt{3}, 20)$.

5. $(10.0, 200.0°) = (10\cos 200°, 10\sin 200°) \approx (-9.4, -3.4)$

9. $(6, -45°) = (6\cos(-45°), 6\sin(-45°)) \approx$
$(6\cos 45°, -6\sin 45°)$ cosine is an even function, sine is an odd function.
$= (6\cdot\frac{\sqrt{2}}{2}, -6\cdot\frac{\sqrt{2}}{2}) = (3\sqrt{2}, -3\sqrt{2})$.

13. $(3.0.\ 4.0)$: $|A| = \sqrt{3^2+4^2} = 5$, $\theta' = \tan^{-1}\frac{4}{3} \approx 53.13°$
$(5, 53.1°)$

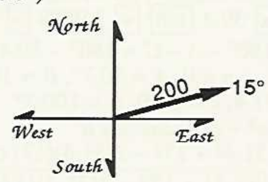
17. $(\sqrt{3}, -2)$; $|A| = \sqrt{(\sqrt{3})^2 + (-2)^2} \approx 2.646$, $\theta' = \tan^{-1}\frac{-2}{\sqrt{3}} \approx -49.11°$; $V_x > 0$ so $\theta = \theta'$: $(2.6, -49.1°)$

21. $(-6.8, 3.4)$; $|A| = \sqrt{(-6.8)^2 + 3.4^2} \approx 7.60$, $\theta' \approx -26.57°$; $\theta \approx -26.57° + 180° \approx 153.43°$; $(7.6, 153.4°)$

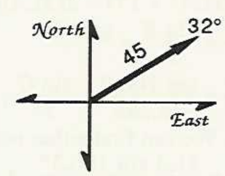
25. $(-3, 8), (2, 12)$ $\quad(-3+2, 8+12) = (-1, 20)$

29.	(30.0, 30°)	= (30 cos 30°, 30 sin 30°)	= (25.980,	15.000)	
	(15.2, 33.6°)	= (15.2 cos 33.6°, 15.2 sin 33.6°)	= (12.660,	8.412)	
			(38.641,	23.412)	= (45.2, 31.2°)
33.	(3.2, –45.0°)	= (3.2 cos(–45°), 3.2 sin(–45°)	= (2.263,	–2.263)	
	(5.9, –59.2°)	= (5.9 cos(–59.2°), 5.9 sin(–59.2°))	= (3.021,	–5.068)	
			(5.284,	–7.331)	= (9.0, –54.2°)
37.	(3.5, 19.2°)	= (3.5 cos 19.2°, 3.5 sin 19.2°)	= (3.305,	1.151)	
	(2.7, 83.1°)	= (2.7 cos 83.1°, 2.7 sin 83.1°)	= (0.324,	2.680)	
	(4.3, 145.7°)	= (4.3 cos 145.7°, 4.3 sin 145.7°)	= (–3.552,	2.423)	
			= (0.077,	6.255)	= (6.3, 89.3°)
41.	(3.5, –25°)	= (3.5 cos(–25°), 3.5 sin(–25°))	= (3.172,	–1.479)	
	(6.8, 25°)	= (6.8 cos 25°, 6.8 sin 25°)	= (6.163,	2.874)	
	(4.2, 50°)	= (4.2 cos 50°, 4.2 sin 50°)	= (2.700,	3.217)	
			= (12.035,	4.612)	= (12.9, 21.0°)

45. $V = (200, 15°)$; $V_x = 200\cos 15° \approx 193$; $V_y = 200\sin 15° \approx 52$.
The aircraft is flying east at 193 knots and north at 52 knots.



49. 18 knots (18 nautical miles per hour) x 2.5 hours = 45 nm (nautical miles).
$V = (45, 32°)$; $V_x = 45\cos 32° \approx 38$ nm; distance east of the harbor (part b)
$V_y = 45\sin 32° \approx 24$ nm; distance north of the harbor (part a).



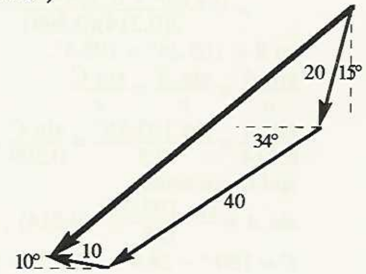
53.

Force V	Horizontal Component V_x	Vertical Component V_y	
(1000, 15°)	966	259	Part (a).
(2000, 15°)	1932	518	Part (b); yes, the components double in value.
(1000, 30°)	866	500	Part (c); no, the components do not double.

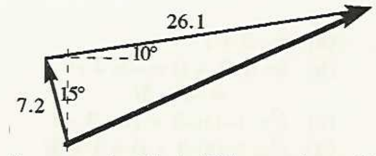
57.

(2.6, 18.3°)	= (2.6 cos 18.3°, 2.6 sin 18.3°)	≈ (2.47,	0.82)
(15.8, –86.2°)	= (15.8 cos(–86.2°, 15.8 sin(–86.2°)	≈ (1.05,	–15.77)
		≈ (3.52,	–14.95) ≈ (15.4, –76.8°)

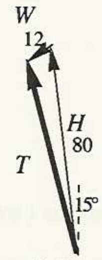
61. $(20, 255°) + (40, 214°) + (10, 170°) \approx (62.6, -140.3°)$.
Thus the ship is about 63 nautical miles from its starting position, at an angle of 40° south of west.



65. $(26.1, 10°) + (7.2, 105°) \approx (26.5, 25.7°)$
Thus the ship is traveling at 26.5 knots in a direction 25.7° north of east.

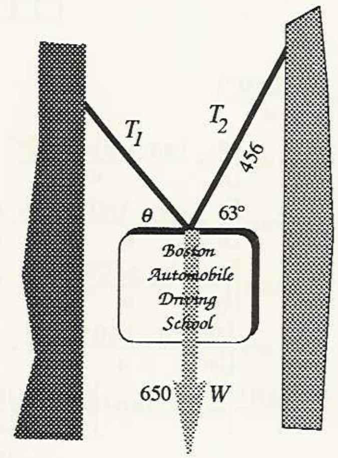


69. $H + W = T$, so $H = T - W$
$= (80, 105°) - (12, 225°)$
$= (80, 105°) + (12, 225° - 180°)$
$= (80, 105°) + (12, 45°)$
$= (-20.71, 77.27) + (8.49, 8.49)$
$= (-12.22, 85.76) \approx (87, 98°)$.
Thus the heading of the aircraft is 8° west of north, and its airspeed is 87 knots.



73. The sign is stationary, so the forces acting on it are balanced (they add to zero).
$T_1 + T_2 + W = 0$
$T_1 = -T_2 - W$
$= -(456, 63°) - (650, 270°)$
$= (456, 63° + 180°) + (650, 270° - 180°)$
To negate a vector, add or subtract 180° from its direction angle.
$= (456, 243°) + (650, 90°)$
$\approx (-207.02, -406.3) + (0, 650)$ Convert to rectangular form.
$\approx (-207.02, 243.7)$
$\approx (320, 130°)$ Convert back to polar form.

Thus the tension in the second cable is 320 pounds, and it makes an angle θ of 50° (180° – 130°) with the horizontal.



Exercise 8–4

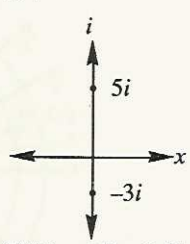
Remember: Given vector $z = a + bi = r \text{ cis } \theta$. Then
$r = \sqrt{a^2 + b^2}$, $\tan\theta = \tan^{-1}\frac{b}{a}$, and
$$\theta = \begin{cases} \theta' & \text{if } a > 0 \\ \theta' - 180° \text{ if } \theta' > 0 & \text{if } a < 0 \\ \theta' + 180° \text{ if } \theta' < 0 & \end{cases}$$

1. $5 - 2i$
$a = 5, b = -2$; $r = \sqrt{a^2 + b^2} = \sqrt{5^2 + (-2)^2} = \sqrt{29} \approx 5.4$.
$\theta' = \tan^{-1}\frac{b}{a} = \tan^{-1}\frac{-2}{5} \approx -21.8°$.
Since $a > 0$ $\theta = \theta'$. Thus the point is 5.4 cis(–21.8°).

5. $-3 + 4i$ $\quad r = \sqrt{(-3)^2 + 4^2} = 5$
$\theta' = \tan^{-1}\frac{4}{-3} \approx -53.1°$. $a < 0$, $\theta' < 0$ so $\theta \approx -53.1° + 180° \approx 126.9°$. The point is 5 cis 126.9°.

9. $3 + 3i$ $\quad r = \sqrt{3^2 + 3^2} = 3\sqrt{2}$.
$\theta' = \tan^{-1} 1 = 45°$; the point is $3\sqrt{2}$ cis 45°.

13. $5i$ Plotting the point shows that $r = 5$ and $\theta = 90°$.
Thus the point is 5 cis 90°.



17. 4.5 cis 35° $\quad 4.5 \cos 35° + (4.5 \sin 35°)i = 3.7 + 2.6i$

21. 13.6 cis (–25°) $\quad 13.6 \cos(-25°) + (13.6 \sin(-25°))i$
$= 12.3 - 5.7i$

25. 10 cis 300° $\quad 10 \cos 300° + (10 \sin 300°)i$
$= 10 \cdot \frac{1}{2} + 10(-\frac{\sqrt{3}}{2})\, i = 5 - 5\sqrt{3}\, i$

29. $\sqrt{8}$ cis 315° $\quad \sqrt{8} \cos 315° + (\sqrt{8} \sin 315°)i$
$= \sqrt{8} \cdot \frac{\sqrt{2}}{2} + \sqrt{8} \cdot (-\frac{\sqrt{2}}{2})i = 2 - 2i$

33. $(5.4 \text{ cis } 300°)(2 \text{ cis } 300°) = 10.8 \text{ cis}(600°)$
$= 10.8 \text{ cis}(600° - 2 \cdot 360°) = 10.8 \text{ cis}(-120°)°$

37. $\dfrac{40 \text{ cis } 80°}{18 \text{ cis } 160°} = \frac{40}{18} \text{cis}(80° - 160°) = \frac{20}{9} \text{cis}(-80°)$

41. $(3 \text{ cis } 200°)^3 = 3^3 \text{ cis}(3 \cdot 200°) = 27 \text{ cis } 600°$
$= 27 \text{ cis}(600° - 2 \cdot 360°) = 27 \text{ cis}(-120°)$

Use: $(r \text{ cis } \theta)^{\frac{1}{n}} = r^{\frac{1}{n}} \text{cis}\left(\frac{\theta}{n} + \frac{k \cdot 360°}{n}\right)$, $0 \le k < n$.

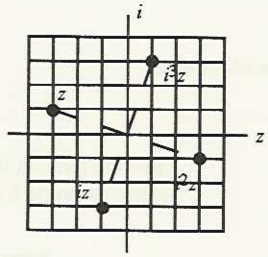
45. Find the 3 cube roots of 8 in exact form.
$8 = 8 \text{ cis } 0°$
Evaluate $8^{\frac{1}{3}} \text{cis}\left(\frac{0°}{3} + \frac{k \cdot 360°}{3}\right) = 2 \text{ cis } (k \cdot 120°)$ for $k = 0, 1, 2$.
$k = 0$: $2 \text{ cis } 0° = 2 \cos 0° + 2 \sin 0° i = 2 + 0i = 2$
$k = 1$: $2 \text{ cis } 120° = 2 \cos 120° + 2 \sin 120° i$
$= 2\left(-\frac{1}{2}\right) + 2\left(\frac{\sqrt{3}}{2}\right)i = -1 + \sqrt{3}i$
$k = 2$: $2 \text{ cis } 240° = 2 \cos 240° + 2 \sin 240° i$
$= 2\left(-\frac{1}{2}\right) + 2\left(-\frac{\sqrt{3}}{2}\right)i = -1 - \sqrt{3}i$
Thus the three cube roots of 8 are $2, -1 + \sqrt{3}\, i, -1 - \sqrt{3}\, i$.

49. Find the 3 cube roots of $75 - 100i$ to the nearest tenth.
$75 - 100i \approx 125 \text{ cis}(306.8699°)$; $(75 - 100i)^{1/3} \approx (125 \text{ cis}(306.8699°))^{1/3}$
Evaluate $\sqrt[3]{125} \text{ cis}\left(\frac{306.8699°}{3} + \frac{k \cdot 360°}{3}\right)$
$= 5 \text{ cis}(102.29° + k \cdot 120°)$ for $k = 0, 1, 2$.
$k = 0$: $5 \text{ cis}(102.29°) = 5 \cos 102.29° + 5 \sin 102.29°\, i$
$\approx -1.1 + 4.9\, i$
$k = 1$: $5 \text{ cis}(102.29° + 120°) = 5 \cos 222.29° + 5 \sin 222.29°\, i$
$\approx -3.7 - 3.4i$
$k = 2$: $5 \text{ cis}(102.29° + 240°) = 5 \cos 342.29° + 5 \sin 342.29°\, i$
$\approx 4.8 - 1.5i$

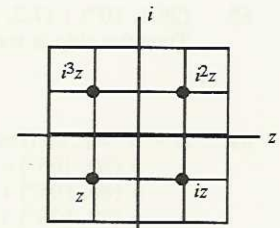
53. $I = \frac{V}{Z}$, so $V = IZ = (10 \text{ cis } 15°)(5 \text{ cis } 30°) = 50 \text{ cis } 45°$.

57. $P = 5 + 2i \approx 5.385 \text{ cis } 21.801°$
$Z = 1 - 4i \approx 4.123 \text{ cis } 284.036°$
$\sqrt{\frac{P}{Z}} \approx \sqrt{\frac{5.385 \text{ cis } 21.801°}{4.123 \text{ cis } 284.036°}} \approx (1.3061 \text{ cis}(-262.235°))^{1/2} \approx$
$\sqrt{1.3061} \text{cis}\left(\frac{-262.235°}{2}\right) \approx 1.143 \text{ cis}(-131.118°) \approx 0.75 + 0.86i$

61. (a) z: $-3 + i$
(b) iz: $i(-3 + i) = -3i + i^2$
$= -1 - 3i$
(c) i^2z: $(-1)(-3 + i) = 3 - i$
(d) i^3z: $(-i)(-3 + i) = 1 + 3i$



65. (a) z $\quad -1 - i$
(b) iz $\quad i(-1 - i) = -i - i^2$
$= 1 - i$
(c) i^2z $\quad (-1)(-1 - i) = 1 + i$
(d) i^3z $\quad (-i)(-1 - i) = -1 + i$



69. $r^{\frac{1}{n}} \operatorname{cis}\left(\frac{\theta}{n} + \frac{k \cdot 360°}{n}\right)$

$$= r^{\frac{1}{n}} \operatorname{cis}\left[\frac{\theta}{n} + \frac{(an + b) \cdot 360°}{n}\right]$$ Replace k by $an + b$.

$$= r^{\frac{1}{n}} \operatorname{cis}\left[\frac{\theta}{n} + \frac{an \cdot 360°}{n} + \frac{b \cdot 360°}{n}\right] = r^{\frac{1}{n}} \operatorname{cis}\left[\frac{\theta}{n} + a \cdot 360° + \frac{b \cdot 360°}{n}\right] = r^{\frac{1}{n}} \operatorname{cis}\left[\left(\frac{\theta}{n} + \frac{b \cdot 360°}{n}\right) + a \cdot 360°\right]$$

$$= r^{\frac{1}{n}} \operatorname{cis}\left[\left(\frac{\theta}{n} + \frac{b \cdot 360°}{n}\right) + a \cdot 360°\right]$$

$$= r^{\frac{1}{n}} \cos\left[\left(\frac{\theta}{n} + \frac{b \cdot 360°}{n}\right) + a \cdot 360°\right] + i\, r^{\frac{1}{n}} \sin\left[\left(\frac{\theta}{n} + \frac{b \cdot 360°}{n}\right) + a \cdot 360°\right]$$

$$\cos\left[\left(\frac{\theta}{n} + \frac{b \cdot 360°}{n}\right) + a \cdot 360°\right] = \cos\left(\frac{\theta}{n} + \frac{b \cdot 360°}{n}\right)\cos(a \cdot 360°) - \sin\left(\frac{\theta}{n} + \frac{b \cdot 360°}{n}\right)\sin(a \cdot 360°)$$

$$= \cos\left(\frac{\theta}{n} + \frac{b \cdot 360°}{n}\right),$$

because $\cos(a{\cdot}360°) = 1$ and $\sin(a{\cdot}360°) = 0$, when a is an integer.

$$\sin\left[\left(\frac{\theta}{n} + \frac{b \cdot 360°}{n}\right) + a \cdot 360°\right] = \sin\left(\frac{\theta}{n} + \frac{b \cdot 360°}{n}\right)\cos(a \cdot 360°) + \cos\left(\frac{\theta}{n} + \frac{b \cdot 360°}{n}\right)\sin(a \cdot 360°)$$

$$= \sin\left(\frac{\theta}{n} + \frac{b \cdot 360°}{n}\right),$$

because $\cos(a{\cdot}360°) = 1$ and $\sin(a{\cdot}360°) = 0$, when a is an integer.

Thus,

$$r^{\frac{1}{n}} \cos\left[\left(\frac{\theta}{n} + \frac{b \cdot 360°}{n}\right) + a \cdot 360°\right] + i\, r^{\frac{1}{n}} \sin\left[\left(\frac{\theta}{n} + \frac{b \cdot 360°}{n}\right) + a \cdot 360°\right]$$

$$= r^{\frac{1}{n}} \cos\left(\frac{\theta}{n} + \frac{b \cdot 360°}{n}\right) + i\, r^{\frac{1}{n}} \sin\left(\frac{\theta}{n} + \frac{b \cdot 360°}{n}\right)$$

$$= r^{\frac{1}{n}} \operatorname{cis}\left(\frac{\theta}{n} + \frac{b \cdot 360°}{n}\right), \text{ where } b < n.$$

This last expression is one of the previous roots.

Exercise 8–5

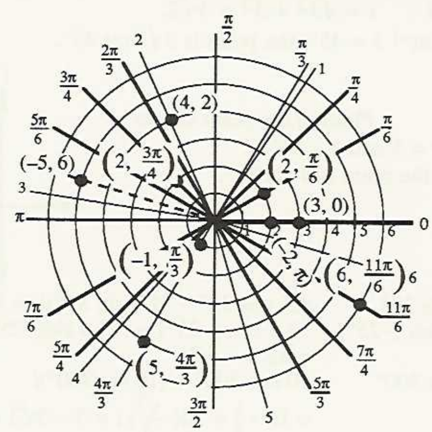
The points for problems 1 through 19 are plotted in the figure. Some special considerations are discussed here.

9. $(-2, \pi) = (2, \pi - \pi) = (2, 0)$

11. $\left(-1, \frac{\pi}{3}\right) = \left(1, \frac{\pi}{3} + \pi\right) = \left(1, \frac{4\pi}{3}\right)$

13. $(4, 2)$; Note: $2 = 2{\cdot}\frac{180°}{\pi} \approx 115°$ (for plotting)

15. $(-5, 6) = (5, 6 - \pi)$; $6 - \pi = (6 - \pi)\left(\frac{180°}{\pi}\right) \approx 164°$. To plot $(-5, 6)$, plot a point 5 units from the center, at an angle about 164°.



Solutions to 1,3,5,7,9,11,13,15,17.

Many answers are possible. To change the sign of r add an odd multiple of π to θ; for the rest add an even multiple of π.

21. $\left(6, \frac{11\pi}{6}\right)$ $\quad \left(-6, \frac{11\pi}{6} - \pi\right)$, $\quad \left(6, \frac{11\pi}{6} + 2\pi\right)$ $\quad \left(6, \frac{11\pi}{6} + 4\pi\right)$
$\left(-6, \frac{5\pi}{6}\right)$ $\quad \left(6, \frac{23\pi}{6}\right)$ $\quad \left(6, \frac{35\pi}{6}\right)$

Use $(r, \theta) = (r \cos \theta, r \sin \theta) = (x, y)$.

25. $\left(4, \frac{\pi}{2}\right) = \left(4 \cos \frac{\pi}{2}, 4 \sin \frac{\pi}{2}\right) = (4{\cdot}0, 4{\cdot}1) = (0, 4)$

29. $\left(4, \frac{4\pi}{3}\right) = \left(4 \cos \frac{4\pi}{3}, 4 \sin \frac{4\pi}{3}\right) = \left(4{\cdot}\left(-\frac{1}{2}\right), 4{\cdot}\left(-\frac{\sqrt{3}}{2}\right)\right) =$
$(-2, -2\sqrt{3})$

33. $(3, 0.82) = (3 \cos 0.82, 3 \sin 0.82) \approx (2.05, 2.19)$

37. $(-2\sqrt{3}, -2)$ $\quad r = \sqrt{(2\sqrt{3})^2 + 2^2} = \sqrt{4 \cdot 3 + 4} = 4.$

$\theta' = \tan^{-1}\left(\frac{-2}{-2\sqrt{3}}\right) = \tan^{-1}\frac{\sqrt{3}}{3} = \frac{\pi}{6}.$

Since $x = -2\sqrt{3} < 0$, and $\theta' > 0$, $\theta = \theta' - \pi = \frac{\pi}{6} - \pi = -\frac{5\pi}{6}.$

Thus the polar coordinates are $(4, -\frac{5\pi}{6})$.

41. $(-4, -4)$ $\quad r = \sqrt{4^2 + 4^2} = 4\sqrt{2}$

$\theta' = \tan^{-1}\left(\frac{-4}{-4}\right) = \tan^{-1} 1 = \frac{\pi}{4}.$

Since $x < 0$, $\theta' > 0$, $\theta = \theta' - \pi = \frac{\pi}{4} - \pi = -\frac{3\pi}{4}$

The point is $\left(4\sqrt{2}, -\frac{3\pi}{4}\right)$ in polar coordinates.

45. $(1, -4)$ $\quad r = \sqrt{1^2 + 4^2} = \sqrt{17} \approx 4.12$

$\theta' = \tan^{-1}(-4) \approx -1.326$; $x > 0$ so $\theta = \theta'$.

$(4.12, -1.33)$

Use $x = r \cos \theta$, $y = r \sin \theta$.

49. $y = 4x$

$r \sin \theta = 4r \cos \theta$

$\sin \theta = 4 \cos \theta$

$\frac{\sin \theta}{\cos \theta} = 4$

$\tan \theta = 4$

53. $y = mx + b, b \neq 0$

$r \sin \theta = mr \cos \theta + b$

$r \sin \theta - mr \cos \theta = b$

$r(\sin \theta - m \cos \theta) = b$

$r = \frac{b}{\sin \theta - m \cos \theta}$

57. $3x^2 + 2y^2 = 1$

$3(r \cos \theta)^2 + 2(r \sin \theta)^2 = 1$

$3r^2 \cos^2\theta + 2r^2 \sin^2\theta = 1$

$r^2(3 \cos^2\theta + 2 \sin^2\theta) = 1$

$r^2 = \frac{1}{3 \cos^2\theta + 2 \sin^2\theta}$

Use $\sin \theta = \frac{y}{r}$; $\cos \theta = \frac{x}{r}$; $r^2 = x^2 + y^2$.

61. $r = 2 \sec \theta$

$r = \frac{2}{\cos \theta}$

$r = \frac{2}{\frac{x}{r}}$

$r = \frac{2r}{x}$

$rx = 2r$

$x = 2$

Assumes $r \neq 0$. It is not since $2 \sec \theta$ is never 0.

65. $r^2 = \sin 2\theta$

$r^2 = 2 \sin \theta \cos \theta$

$r^2 = 2\frac{y}{r} \cdot \frac{x}{r}$

$r^4 = 2xy$

$(x^2 + y^2)^2 = 2xy$

$x^4 + 2x^2y^2 - 2xy + y^4 = 0$

69. $r = \frac{3}{1 - 2 \sin \theta}$

$r(1 - 2 \sin \theta) = 3$

$r(1 - \frac{2y}{r}) = 3$

$r - 2y = 3$

$r = 2y + 3$

$r^2 = (2y + 3)^2$

$x^2 + y^2 = 4y^2 + 12y + 9$

$0 = 3y^2 + 12y - x^2 + 9$

73. Consider a point $P = (r, \theta)$, where $r < 0$. Then $P = (-r, \theta + \pi)$, where $-r > 0$. Therefore, since $-r > 0$, $y = -r \sin(\theta + \pi)$ is true.

$y = -r \sin(\theta + \pi)$ $\quad$ A true statement.

$= -r(\sin \theta \cos \pi + \cos \theta \sin \pi)$ $\quad$ $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta.$

$= -r(\sin \theta (-1) + \cos \theta (0)) = -r(-\sin \theta) = r \sin \theta.$

Thus $y = r \sin \theta$, even if $r < 0$.

77. $r = 1 + 2 \sin 2\theta$

$r = 1 + 2(2 \sin \theta \cos \theta)$

$r = 1 + 4\frac{y}{r} \cdot \frac{x}{r}$

$r^3 = r^2 + 4xy$

$r^6 = (r^2 + 4xy)^2$

$(r^2)^3 = (x^2 + y^2 + 4xy)^2$

$(x^2 + y^2)^3 = (x^2 + y^2 + 4xy)^2$

Chapter 8 Review

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

1. $a = 10.6, A = 47.9°, B = 10.3°$

$C = 180° - 47.9° - 10.3° = 121.8°$

$\frac{\sin 47.9°}{10.6} = \frac{\sin 10.3°}{b} = \frac{\sin 121.8°}{c}$

$b = \frac{10.6 \sin 10.3°}{\sin 47.9°} \approx 2.6$

$c = \frac{10.6 \sin 121.8°}{\sin 47.9°} \approx 12.1$

3. $a = 10.0, b = 13.0, B = 79.0°$

$\frac{\sin A}{10} = \frac{\sin 79°}{13} = \frac{\sin C}{c}$

Observe that A must be acute, since side a is not the longest side of the triangle.

$\sin A = \frac{10 \sin 79°}{13}$, so $A \approx 49.03° \approx 49.0°$

$C \approx 180° - 79° - 49.03° \approx 51.97° \approx 52.0°.$

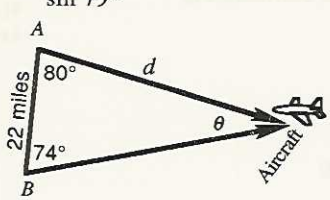
$\frac{\sin 79°}{13} \approx \frac{\sin 51.97°}{c}$, so $c \approx \frac{13 \sin 51.97°}{\sin 79°} \approx 10.4$

5. $\theta = 180° - 80° - 74°$

$= 26°$; $\frac{\sin 26°}{22} = \frac{\sin 74°}{d}$

$d = \frac{22 \sin 74°}{\sin 26°}$

$d \approx 48$ miles



7. $b = 60.0, c = 20.0, A = 92.1°$

$a^2 = b^2 + c^2 - 2bc \cos A$

$a^2 = 60^2 + 20^2 - 2(60)(20) \cos 92.1°$

$a \approx 63.937 \approx 63.9$

$\frac{\sin 92.1°}{63.937} = \frac{\sin B}{60} = \frac{\sin C}{20}$

Neither B nor C can be obtuse, since angle A must be the largest angle in the triangle.

$\sin B = \frac{60 \sin 92.1°}{63.937}$, so $B \approx 69.7°$

$C \approx 180° - 69.7° - 92.1° \approx 18.2°$

9. $a = 43.5, b = 17.8, c = 35.0$

Find the largest angle first, since the two smallest angles must be acute, and the law of cosines does not have an ambiguous case. The largest angle is opposite the longest side, so it is A.

$a^2 = b^2 + c^2 - 2bc \cos A$

$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \approx \frac{17.8^2 + 35^2 - 43.5^2}{2(17.8)(35)}$

$A \approx 106.335° \approx 106.3°$

$\frac{\sin 106.335°}{43.5} = \frac{\sin B}{17.8} = \frac{\sin C}{35}$

$\sin B = \frac{17.8 \sin 106.335°}{53.5}$

$B \approx 23.123° \approx 23.1°$

$C \approx 180° - 23.123° - 106.335° \approx 50.542° \approx 50.5°$

11. $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$a = \sqrt{(4 - 5)^2 + (7 - 2)^2} = \sqrt{26}$

$b = \sqrt{(5 - (-2))^2 + (2 - 5)^2} = \sqrt{58}$

$c = \sqrt{(4 - (-2))^2 + (7 - 5)^2} = \sqrt{40}$

Side b is the longest side, so we use the law of cosines to find angle B.

$b^2 = a^2 + c^2 - 2ac \cos B$

$58 = 26 + 40 - 2(\sqrt{26})(\sqrt{40}) \cos B$

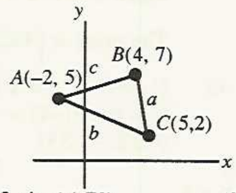
$\cos B = \dfrac{8}{2(\sqrt{26})(\sqrt{40})} \approx 0.1240$, so $B \approx 82.875° \approx 82.9°$

$\dfrac{\sin A}{\sqrt{26}} = \dfrac{\sin 82.875°}{\sqrt{58}}$

$\sin A = \dfrac{\sqrt{26}}{\sqrt{58}} \sin 82.875°$

$A \approx 41.6°$

$C \approx 180° - 82.9° - 41.6° \approx 55.5°$



13. $V = (27.2, 29°)$
$V_x = 27.2 \cos 29° \approx 23.8$
$V_y = 27.2 \sin 29° \approx 13.2$

15. Convert (450,34.6°) to rectangular coordinates.
$(450, 34.6°) = (450 \cos 34.6°, 450 \sin 34.6°) \approx (370, 256)$
Thus the horizontal component is 370 knots, and the vertical component is 256 knots.

17. $(33.0, 14.7°) = (33 \cos 14.7°, 33 \sin 14.7°) \approx (31.92,\ 8.37)$
$(15.2, 33.6°) = (15.2 \cos 33.6°, 15.2 \sin 33.6°) \approx (12.66,\ 8.41)$
$(44.58,\ 16.79) \approx (47.6, 20.6°)$

19. $(3.5, 29.2°) \approx (3.06,\ 1.71)$
$(1.7, 43.1°) \approx (1.24,\ 1.16)$
$(4.3, 115.0°) \approx (-1.82,\ 3.90)$
$\approx (2.48,\ 6.77) \approx (7.2, 69.9°)$

21. $(126, 223°) = (126 \cos 223°, 126 \sin 223°) \approx (-92.15,\ -85.93)$
$(158, 311°) = (158 \cos 311°, 158 \sin 311°) \approx (103.66,\ -119.24)$
$(11.51,\ -205.18) \approx (205, -87°)$

Magnitude ≈ 205 pounds; direction ≈ −87°.

23. $\sqrt{3} + 3i \quad r = \sqrt{(\sqrt{3})^2 + 3^2} = \sqrt{12} = 2\sqrt{3}$
$\theta = \theta' = \tan^{-1} \frac{3}{\sqrt{3}} = \tan^{-1}\sqrt{3} = 60°$; $2\sqrt{3}$ cis $60°$

25. $3 \text{ cis } 35° = 3 \cos 35° + 3 \sin 35° \approx 2.5 + 1.7i$

27. $3 \text{ cis } 240° = 3 \cos 240° + 3i \sin 240° = 3\cdot\left(-\frac{1}{2}\right) + 3i\left(-\frac{\sqrt{3}}{2}\right)$
$= -1.5 - 1.5\sqrt{3}\, i$

29. $(2 \text{ cis } 25°)(3 \text{ cis } 45°) = 2\cdot3 \text{ cis}(25° + 45°) = 6 \text{ cis } 70°$

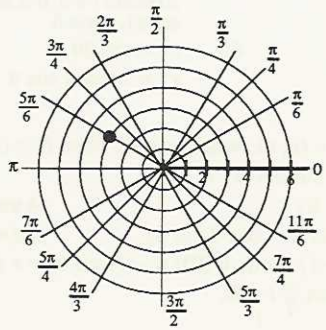
31. $\dfrac{40 \text{ cis } 120°}{5 \text{ cis } 20°} = \dfrac{40}{5} \text{cis}(120° - 20°) = 8 \text{ cis } 100°$

33. $(2 \text{ cis } 130°)^3 = 2^3 \text{ cis}(3\cdot130°) = 8 \text{ cis } 390° =$
$8 \text{ cis}(390° - 360°) = 8 \text{ cis } 30°$

35. $(0.8 + 0.6i)^8 \approx [\text{cis}(36.860°)]^8 \approx \text{cis}(8\cdot36.8699) \approx \text{cis } 294.96°$
$\approx 0.42 - 0.91i$.

37. $6 - 5i \approx \sqrt{61} \text{ cis}(-39.8056°)$
Evaluate $(\sqrt{61})^{1/3} \text{ cis}\left(\dfrac{-39.8056°}{3} + \dfrac{k\cdot360}{3}\right) \approx$
$1.984 \text{ cis}(-13.2685° + k\cdot120°)$ for $k = 0, 1, 2$.
$k = 0$: $1.984 \text{ cis}(-13.2685°) \approx 1.93 - 0.46i$
$k = 1$: $1.984 \text{ cis}(-13.2685° + 120°) \approx 1.984 \text{ cis}(106.7315°) \approx$
$-0.57 + 1.90i$
$k = 2$: $1.984 \text{ cis}(-13.2685° + 240°) \approx 1.984 \text{ cis}(226.7315°) \approx$
$-1.36 - 1.44i$

41.



43. $\left(3, \frac{11\pi}{6}\right) = \left(3 \cos \frac{11\pi}{6}, 3 \sin \frac{11\pi}{6}\right) = \left(3\cdot\frac{\sqrt{3}}{2}, 3\left(-\frac{1}{2}\right)\right) = \left(\frac{3}{2}\sqrt{3}, -\frac{3}{2}\right)$

45. $\left(-2, \frac{5\pi}{3}\right) = \left(-2 \cos \frac{5\pi}{3}, -2 \sin \frac{5\pi}{3}\right) = \left(-2\cdot\frac{1}{2}, -2\left(-\frac{\sqrt{3}}{2}\right)\right) = (-1, \sqrt{3})$

47. $(5, 2) \approx (5 \cos 2, 5 \sin 2) \approx (-2.1, 4.5)$

49. $(2, 1) \quad r = \sqrt{2^2 + 1^2} = \sqrt{5} \approx 2.24$
$\theta = \theta' = \tan^{-1} \frac{1}{2} \approx 0.46$; $(2.24, 0.46)$

51. $(-1, -4) \quad r = \sqrt{1^2 + 4^2} = \sqrt{17} \approx 4.12$
$\theta' = \tan^{-1} 4 \approx 1.326$; $x < 0$, $\theta' > 0$, so $\theta = \theta' - \pi \approx 1.326 - \pi$
≈ -1.82 ; $(4.12, -1.82)$

53. $y = 4x + 2$
$r \sin \theta = 4r \cos \theta + 2$
$r \sin \theta - 4r \cos \theta = 2$
$r(\sin \theta - 4 \cos \theta) = 2$
$r = \dfrac{2}{\sin \theta - 4 \cos \theta}$

55. $y^2 - 3x = 0$
$(r \sin \theta)^2 - 3r \cos \theta = 0$
$r^2 \sin^2\theta - 3r \cos \theta = 0$
$r \sin^2\theta - 3 \cos \theta = 0$
$r \sin^2\theta = 3 \cos \theta$
$r = \dfrac{3 \cos\theta}{\sin^2\theta}$
$r = 3\dfrac{\cos \theta}{\sin \theta}\cdot\dfrac{1}{\sin \theta}$
$r = 3 \cot \theta \csc \theta$ Alternate form of answer.

57. $x = 2$
$r \cos \theta = 2$
$r = \dfrac{2}{\cos \theta}$
$r = 2 \sec \theta$ (Alternate form of answer.)

59. $r = 2 \sec \theta$
$r = 2\cdot\dfrac{1}{\cos \theta}$
$r = 2\cdot\dfrac{r}{x}$
$1 = \dfrac{2}{x}$ Divide each member by r.
$x = 2$

61. $r^2 = \tan \theta$
$x^2 + y^2 = \dfrac{y}{x}$
$x^3 + xy^2 = y$
$x^3 + xy^2 - y = 0$

63. $r = \dfrac{3}{2 - \sin \theta}$
$r = \dfrac{3}{2 - \frac{y}{r}}$
$r\left(2 - \frac{y}{r}\right) = 3$
$2r - y = 3$
$2r = y + 3$
$(2r)^2 = (y + 3)^2$
$4r^2 = y^2 + 6y + 9$
$4(x^2 + y^2) = y^2 + 6y + 9$
$4x^2 + 3y^2 - 6y - 9 = 0$

Chapter 8 Test

1. $B = 180° - 13.5° - 82.1° = 84.4°$

$\frac{\sin 13.5°}{a} = \frac{\sin 84.4°}{22.6} = \frac{\sin 82.1°}{c}$

$a = \frac{22.6 \sin 13.5°}{\sin 84.4°} \approx 5.3$

$c = \frac{22.6 \sin 82.1°}{\sin 84.4°} \approx 22.5$

3. $b^2 = a^2 + c^2 - 2ac \cos B$

$b^2 = 25.9^2 + 16.2^2 - 2(25.9)(16.2) \cos 100° \approx 1078.97$

$b \approx 32.848 \approx 32.8$

$\frac{\sin A}{a} = \frac{\sin B}{b}$

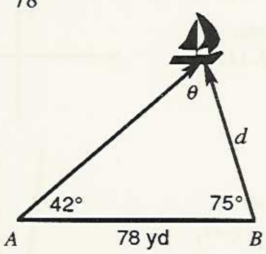
$\frac{\sin A}{25.9} = \frac{\sin 100°}{32.848}$

$\sin A = \frac{25.9 \sin 100°}{32.848}$; $A \approx 50.9°$

$C \approx 180° - 50.9° - 100° \approx 29.1°$

5. $\theta = 180° - 42° - 75° = 63°$

$\frac{\sin 42°}{d} = \frac{\sin 63°}{78}$; $d \approx 59$ yards



7. $(2, 30°) = (2 \cos 30°, 2 \sin 30°)$

$= \left(2 \cdot \frac{\sqrt{3}}{2}, 2 \cdot \frac{1}{2}\right) = (\sqrt{3}, 1)$

9. Add the vectors (5.4, 19.0°) and (8.0, 123°). Round the results to the nearest tenth.

$(5.4, 19°) = (5.4 \cos 19°, 5.4 \sin 19°) \approx (5.11, \ 1.76)$

$(8, 123°) = (8 \cos 123°, 8 \sin 123°) \approx (-4.36, \ 6.71)$

$(0.75, \ 8.47)$

$\approx (8.5, 84.9°)$

11. $4 - 5i \qquad r = \sqrt{4^2 + 5^2} = \sqrt{41} \approx 6.40$

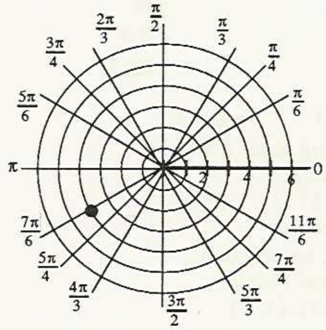
$\theta = \theta' - \tan^{-1}(-\frac{5}{4}) \approx -51.3 \qquad \theta = \theta'$ because $a > 0$

$6.40 \operatorname{cis}(-51.3)$

13. $(2 \operatorname{cis} 325°)(7 \operatorname{cis} 145°) = 2 \cdot 7 \operatorname{cis} (325° + 145°) = 14 \operatorname{cis} 470° = 14 \operatorname{cis}(470° - 360°) = 14 \operatorname{cis} 110°$.

15. $(3 \operatorname{cis} 150°)^3 = 3^3 \operatorname{cis}(3 \cdot 150°) = 27 \operatorname{cis} 450° = 27 \operatorname{cis}(450° - 360°) = 27 \operatorname{cis} 90°$.

17. $(-4, \frac{\pi}{6}) = (4, \frac{\pi}{6} + \pi) = (2, \frac{7\pi}{6})$



19. $(-\sqrt{3}, -1)$

$r = \sqrt{(\sqrt{3})^2 + 1^2} = 2$

$\theta' = \tan^{-1} \frac{1}{\sqrt{3}} = \frac{\pi}{6}$

Since $x < 0$, $\theta' > 0$, $\theta = \theta' - \pi = \frac{\pi}{6} - \pi$

$= -\frac{5\pi}{6}$; $(2, -\frac{5\pi}{6})$

21. $2y^2 - x = 5$

$2(r \sin \theta)^2 - r \cos \theta = 5$

$2r^2 \sin^2\theta - r \cos \theta = 5$

$2r^2 \sin^2\theta - r \cos \theta - 5 = 0$

23. $r^2 = \cos 2\theta$

$x^2 + y^2 = \cos^2\theta - \sin^2\theta$

$x^2 + y^2 = \frac{x^2}{r^2} - \frac{y^2}{r^2}$

$x^2 + y^2 = \frac{x^2 + y^2}{r^2}$

$x^2 + y^2 = \frac{x^2 - y^2}{x^2 + y^2}$

$(x^2 + y^2)^2 = x^2 - y^2$

$(x^2 + y^2)^2 - x^2 + y^2 = 0$

Exercise 9–1

1. $f(x) = b^x$, $b > 0$ and $b \neq 1$

5. $7^{\pi} \cdot 7^{3\pi}$
$7^{\pi+3\pi}$
$7^{4\pi}$
$(7^4)^{\pi}$
2401^{π}

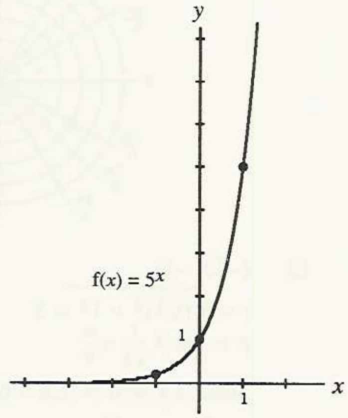
9. $(3^{\sqrt{2}})^{\sqrt{2}}$
3^2
9

13. $3^x = \sqrt{27}$
$3^x = \sqrt{3^3}$
$3^x = 3^{3/2}$
$x = \frac{3}{2}$

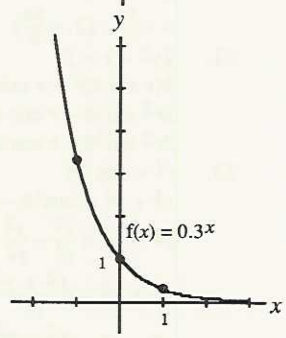
17. $2^x = \frac{1}{8}$
$2^x = 2^{-3}$
$x = -3$

21. $3^x = \sqrt[3]{243}$
$3^x = \sqrt[3]{3^5}$
$3^x = 3^{5/3}$
$x = \frac{5}{3}$

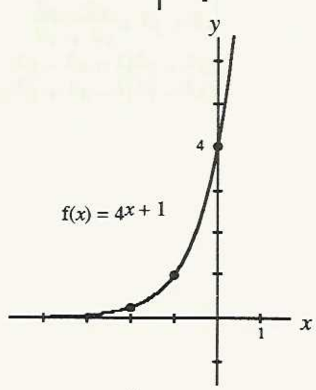
25. $f(x) = 5^x$ $\quad b = 5$
Increasing since $b > 1$.
y–intercept:
$f(0) = 5^0 = 1$ $\quad (0, 1)$
x–intercept:
$0 = 5^x$; no solution.
Additional Points:
$(-1, 0.2)$, $(1, 5)$



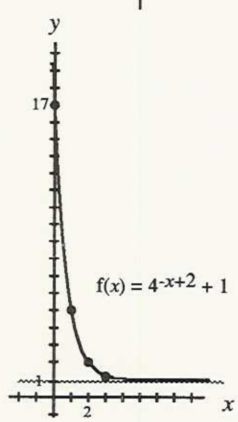
29. $f(x) = 0.3^x$ $\quad b = 0.3$
Decreasing since $b < 1$.
y–intercept:
$f(0) = 0.3^0 = 1.$ $\quad (0, 1)$
x–intercept:
$0 = 0.3^x$; no solution.
Additional Points:
$(-1, 3.3)$, $(1, 0.3)$



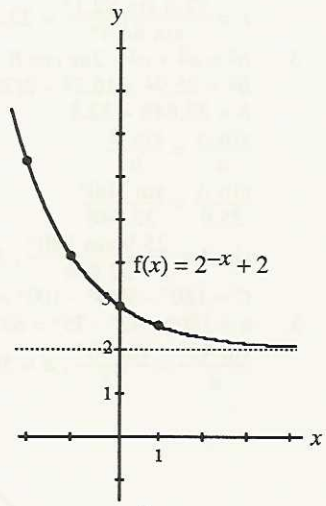
33. $f(x) = 4^{x+1} = 4^1 \cdot 4^x$
$= 4(4^x)$ $\quad b = 4$
This is the function $y = 4^x$ with a vertical scaling factor of 4.
Increasing since $b > 1$.
y–intercept:
$f(0) = 4^1 = 4$ $\quad (0, 4)$
x–intercept: $0 = 4^{x+1}$
$0 = 4 \cdot 4^x$
$0 = 4^x$
no solution
Additional Points:
$(-2, 0.25)$, $(-1, 1)$



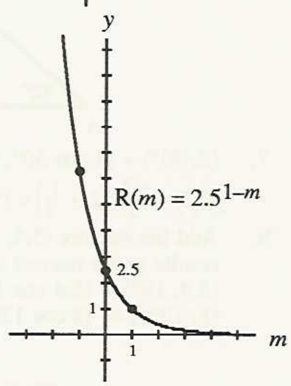
37. $f(x) = 4^{-x+2} + 1$
$= 4^2 \cdot 4^{-x} + 1$
$= 16 \cdot (4^{-1})^x + 1$
$= 16(\frac{1}{4})^x + 1$ $\quad b = \frac{1}{4}$
Decreasing since $b < 1$.
This is the graph of $y = (\frac{1}{4})^x$ with a vertical scaling factor of 16 and shifted up one unit.
y-intercept:
$f(0) = 2^2 + 1 = 17$ $\quad (17, 0)$
x-intercept: $0 = 16(\frac{1}{4})^x + 1$
$-1 = 16(\frac{1}{4})^x$
no solution as the left member is negative and the right is non negative.
Additional Points: $\quad (1, 5)$, $(2, 2)$, $(3, 1.25)$



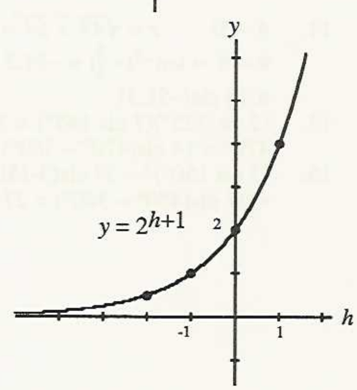
41. $f(x) = 2^{-x} + 2$
$= (\frac{1}{2})^x + 2;$ $\quad b = \frac{1}{2}$
This is $y = (\frac{1}{2})^x$ but shifted up two units; decreasing since $b < 1$.
y-intercept:
$f(0) = 2^0 + 2 = 3$ $\quad (0, 3)$
x-intercept:
$0 = (\frac{1}{2})^x + 2$
$-2 = (\frac{1}{2})^x$;
no solution since
$-2 < 0, (\frac{1}{2})^x > 0.$
Additional Points:
$(-2,6)$, $(-1,4)$, $(1,2.5)$



45. $R(m) = 2.5^{1-m}$
$= 2.5^1(2.5^{-m})$
$= 2.5(\frac{1}{2.5})^m$
$= 2.5(0.4)^m;$ $b = 0.4$
Additional Points: $(-1, 6.25)$, $(0, 2.5)$, $(1, 1)$



49. $f(h) = 2^{h+1}$
$= 2(2^h)$ $\quad b = 2$
Additional Points:
$(-2, 0.5)$, $(-1, 1)$, $(0, 1)$, $(1, 4)$



53. In inches the radius of the earth is
4000 miles x 5280 $\frac{\text{feet}}{\text{mile}}$ x 12 $\frac{\text{inches}}{\text{foot}} \approx 4(10^3)(5)(10^3)(10) = 20(10^7) = 2(10^8)$ inches.
$V = \frac{4}{3}\pi r^3$, $r \approx 2(10^8)$ inches, so $V = \frac{4}{3}\pi(2(10^8))^3 \approx \frac{4}{3}(3)(8)(10^{24}) = 32(10^{24})$ cubic inches in the earth. If there are one million (10^6) grains of sand in a cubic inch, then $32(10^{24})(10^6) = 32(10^{30})$ grains of sand would be needed. This is closest to 10^{30} (value (b)).

Exercise 9-2

1. $3^x = 27$
   $3^x = 3^3$
   $x = 3$

5. $3^x = \frac{1}{27}$
   $3^x = 3^{-3}$
   $x = -3$

9. $10^k = 0.1$
   $10^k = 10^{-1}$
   $k = -1$

13. $\log_4 256$ is 4 since 4^4 is 256

17. $5 \log_3 27 = 5(3) = 15$

21. $5(3 \log_2 \frac{1}{8} + 2 \log_{10} 0.1) =$
    $5[3(-3) + 2(-1)] = -55$

25. $2^3 = 8$

29. $12^2 = x + 3$

33. "4 is the logarithm to the base 2 of 16"; $4 = \log_2 16$

37. "y is the logarithm to the base m of $x + 1$";
    $y = \log_m(x + 1)$

41. $\log_2 x = 4$
    $2^4 = x$
    $16 = x$

45. $\log_x 64 = 6$
    $x^6 = 64$
    $x^6 = 2^6$
    $x = 2$

49. $\log_x 0.1 = -1$
    $x^{-1} = 0.1$
    $x^{-1} = 10^{-1}$
    $x = 10$

53. $\log_2 100$
    $2^6 = 64$
    $2^7 = 128$
    so $6 < \log_2 100 < 7$

57. $\log_2 0.3$
    $0.25 < 0.3 < 0.5$
    $\frac{1}{4} < 0.3 < \frac{1}{2}$
    $2^{-2} < 0.3 < 2^{-1}$
    so $-2 < \log_2 0.3 <$ -1

Apply the following rules in problems 59 – 76.

Composition of exponential with logarithm function $\log_b(b^x) = x$.
Composition of logarithm with exponential function $b^{(\log_b x)} = x$.

61. $\log_5 5^1 = 1$

65. $\log_{10} 10^{18} = 18$

69. $5 \log_5 1 = 5 \log_5 5^0 = 5 \cdot 0 = 0$

73. $4^{\log_2 9} = (2^2)^{\log_2 9} = 2^{2\log_2 9} = 2^{\log_2 81} = 81$

77. (a) $2^9 = 512$ and $2^{10} = 1{,}024$ so 10 bits are needed to represent 843
    (b,c) $2^{13} = 8{,}192$ and $2^{14} = 16{,}384$ so 14 bits are needed to represent both 9,400 and 16,000
    (d) $2^{15} = 32\ 768$ and $2^{16} = 65{,}536$ so 16 bits are needed to represent 35,312

81. $\log_b x = y$ if and only if $b^y = x$, $b > 0$ and $b \neq 1$.

Exercise 9-3

1. $\log_2 3x = 3$
   $3x = 2^3$
   $3x = 8$
   $x = \frac{8}{3}$

5. $\log_3 5x = \log_3(2x + 1)$
   $5x = 2x + 1$
   $3x = 1$
   $x = \frac{1}{3}$

9. $\log_2(5x - 1) = -4$
   $5x - 1 = 2^{-4}$
   $5x = 1 + \frac{1}{16} = \frac{17}{16}$
   $x = \frac{17}{80}$

13. $\log_5(x + 1) = \log_{10} 100$
    $\log_5(x + 1) = 2$
    $x + 1 = 5^2$
    $x = 24$

17. $\log_2 5 + \log_2(3 - 2x) = \log_2 6$
    $\log_2[5(3 - 2x)] = \log_2 6$
    $15 - 10x = 6$
    $9 = 10x$
    $x = \frac{9}{10}$.

21. $\log_5 x - \log_5 3 = \log_5 2$
    $\log_5 \frac{x}{3} = \log_5 2$
    $\frac{x}{3} = 2$
    $x = 6$

25. $\log_2 x^4 = 12$
    $x^4 = 2^{12}$
    $x^4 = (2^3)^4$
    $x = \pm 2^3 = \pm 8$. Either solution is valid in the expression $\log_2 x^4$ but the problem states $x > 0$, so the solution is 8.

29. $\log_6(2xy)$
    $\log_6 2 + \log_6 x + \log_6 y$

33. $\log_3 \dfrac{3xy}{2z}$
    $\log_3(3xy) - \log_3(2z)$
    $\log_3 3 + \log_3 x + \log_3 y$
    $- (\log_3 2 + \log_3 z)$
    $1 + \log_3 x + \log_3 y - \log_3 2 - \log_3 z$

37. $\log_2 4x^3y^2z^5$
    $\log_2 4 + \log_2 x^3 + \log_2 y^2 + \log_2 z^5$
    $2 + 3\log_2 x + 2\log_2 y + 5\log_2 z$

41. $\log_a 6$
    $\log_a(2 \cdot 3)$
    $\log_a 2 + \log_a 3$
    $0.3562 + 0.5646$
    0.9208

45. $\log_a 81$
    $\log_a 3^4$
    $4\log_a 3$
    $4 \cdot 0.5646$
    2.2584

49. $\log_a 14 \quad = 1.3562$
    $-\log_a 2 \quad = 0.3562$
    $\log_a 14 - \log_a 2 = 1$
    $\log_a \frac{14}{2} = 1$
    $\log_a 7 = 1$
    $a^1 = 7$
    $a = 7$

53. $\log_a \sqrt[n]{x} = \log_a x^{1/n}$
    $= \frac{1}{n} \log_a x$.

Exercise 9-4

1. 1.7160

5. 1.0253

9. 3.9405

13. 5.2470

17. –5.8091

21. log 90,000,000,000,000,000,000
    log (9×10^{19})
    log 9 + log 10^{19}
    0.9542 + 19
    19.9542

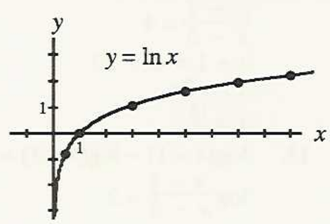
25. log 0.000 000 000 120 004
    log $(1.20004 \times 10^{-10})$
    log 1.20004 + log 10^{-10}
    0.0792 + (–10)
    –9.9208

29. 18 billion =
    18,000,000,000 = 18×10^9
    log (18×10^9)
    log 18 + log 10^9
    1.2553 + 9
    10.2553

33. $\log_8 0.78 = \dfrac{\log 0.78}{\log 8} \approx$
    –0.1195

37. With the $\boxed{\ln}$ key on a calculator it is relatively simple to make a table of values.

x	0.5	1	3	5	7	9
y	0.7	0	1.1	1.6	1.9	2.2

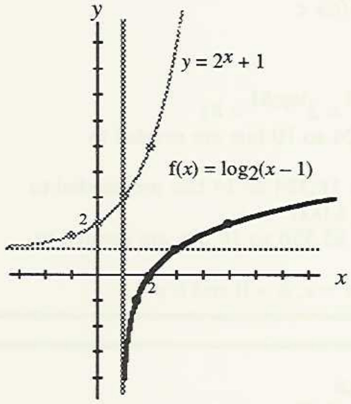


To graph a logarithmic function a) Compute the inverse function (an exponential function), b) Calculate ordered pairs for the inverse function, c) Reverse the components of these ordered pairs to obtain ordered pairs for the logarithmic function, d) Plot all the ordered pairs; sketch both functions.

41. $f(x) = \log_2(x-1)$
$y = \log_2(x-1)$
Compute the inverse function.
$x = \log_2(y-1)$
$y - 1 = 2^x$
$y = 2^x + 1$

Calculated Points

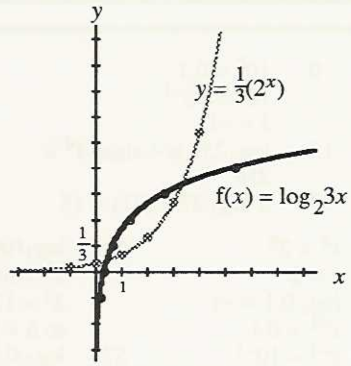
$y = 2^x + 1$	$y = \log_2(x-1)$
(−1, 1.5)	(1.5, −1)
(0, 2)	(2, 0)
(1, 3)	(3, 1)
(2, 5)	(5, 2)



45. $f(x) = \log_2 3x$
$y = \log_2 3x$ Calculate inverse function.
$x = \log_2 3y$
$3y = 2^x$
$y = \frac{1}{3}(2^x)$

Calculated Points

$y = \frac{1}{3}(2^x)$	$y = \log_2 3x$
$(-1, \frac{1}{6})$	$(\frac{1}{6}, -1)$
$(0, \frac{1}{3})$	$(\frac{1}{3}, 0)$
$(1, \frac{2}{3})$	$(\frac{2}{3}, 1)$
$(2, 1\frac{1}{3})$	$(1\frac{1}{3}, 2)$
$(3, 2\frac{2}{3})$	$(2\frac{2}{3}, 3)$
$(4, 5\frac{1}{3})$	$(5\frac{1}{3}, 4)$



49. $10^{-0.33} \approx 0.47$
53. $e^{4.8} \approx 121.51$
57. $\ln 2e^{3x}$
$\ln 2 + \ln e^{3x}$
$3x + \ln 2 \quad \ln e^x = x$
61. $e^{\ln 100}$
$100 \qquad e^{\ln x} = x$
65. $\ln 5e^x$
$\ln 5 + \ln e^x$
$x + \ln 5 \quad \ln e^x = x$

69. $A = P(1 + \frac{i}{n})^{nt}$; $P = 1000$, $i = 0.08$, $n = 12$ (months), $t = 2$ (years)
$A = 1000(1 + \frac{0.08}{12})^{24}$
$A = \$1,172.89$

73. $S = k \log\left(\frac{I}{I_0}\right)$, $k = 12$, and $I = 6I_0$.
$S = 12 \log\left(\frac{6I_0}{I_0}\right) = 12 \log 6 \approx 9.3$

77. $Q = 0.07L\left(\dfrac{T_{\text{in}} - T_{\text{out}}}{\log \dfrac{T_{\text{earth}} - T_{\text{in}}}{T_{\text{earth}} - T_{\text{out}}}}\right)$; $L = 80$, $T_{\text{in}} = 30$, $T_{\text{out}} = 42$, $T_{\text{earth}} = 54$.
$Q = 0.07(80)\dfrac{30 - 42}{\log \dfrac{54 - 30}{54 - 42}} = 5.6\dfrac{-12}{\log \frac{24}{12}}$
$= \dfrac{-67.2}{\log 2} \approx -223$ BTU/hour.

Exercise 9-5

1. $8^x = 32^{3-2x}$
$(2^3)^x = (2^5)^{3-2x}$
$2^{3x} = 2^{15-10x}$
$3x = 15 - 10x$
$x = \frac{15}{13}$

5. $(\sqrt{8})^{2x-2} = 4^{3x}$
$[(2^3)^{1/2}]^{2x-2} = (2^2)^{3x}$
$2^{3x-3} = 2^{6x}$
$3x - 3 = 6x$
$x = -1$

9. $\log(x+1) - \log(x-3) = \log 4$
$\log\frac{x+1}{x-3} = \log 4$
$\frac{x+1}{x-3} = 4$
$x + 1 = 4x - 12$
$13 = 3x$
$x = \frac{13}{3}$

13. $\log(x-1) - \log(x-3) = 2$
$\log\frac{x-1}{x-3} = 2$
$10^2 = \frac{x-1}{x-3}$
$100x - 300 = x - 1$
$x = \frac{299}{99}$

17. $25 = x^4$
$x = \pm\sqrt[4]{25} \approx \pm 2.2$

21. $(x+2)^4 = 200$
$x + 2 = \pm\sqrt[4]{200}$
$x = -2 \pm \sqrt[4]{200} \approx -5.8$
1.8

25. $41^{2x-1} = 2^x$
$\log 41^{2x-1} = \log 2^x$
$(2x-1)\log 41 = x \log 2$
$2x \log 41 - \log 41 = x \log 2$
$2x \log 41 - x \log 2 = \log 41$
$x(2 \log 41 - \log 2) = \log 41$
$x = \dfrac{\log 41}{2 \log 41 - \log 2} \approx 0.6$

29. $5^{x+1} = 3^{x-1}$
$\log 5^{x+1} = \log 3^{x-1}$
$(x+1)\log 5 = (x-1)\log 3$
$x \log 5 + \log 5 = x \log 3 - \log 3$
$\log 5 + \log 3 = x \log 3 - x \log 5$
$\log 5 + \log 3 = x(\log 3 - \log 5)$
$\dfrac{\log 3 + \log 5}{\log 3 - \log 5} = x \approx -5.3$

33. $\log_x 10 = 5$
$x^5 = 10$
$x = \sqrt[5]{10} = 10^{1/5} \approx 1.58$

37. $\log_{2x} 14 = 3$
$(2x)^3 = 14$
$2x = \sqrt[3]{14}$
$x = \frac{\sqrt[3]{14}}{2} \approx 1.21$

41. $\log(\log x) = 3$
$10^3 = \log x$
$1000 = \log x$
$10^{1000} = x$

45. $\log_2 x + \log_3 x = 5$
$\dfrac{\log x}{\log 2} + \dfrac{\log x}{\log 3} = 5$
$\dfrac{\log x}{\log 2}(\log 2)(\log 3) + \dfrac{\log x}{\log 3}(\log 2)(\log 3)$
$= 5(\log 2)(\log 3)$
$(\log 3)(\log x) + (\log 2)(\log x)$
$= 5(\log 2)(\log 3)$
$\log x (\log 3 + \log 2) = 5(\log 2)(\log 3)$
$\log x = \dfrac{5(\log 2)(\log 3)}{\log 3 + \log 2}$
$x = 10^{\frac{5(\log 2)(\log 3)}{\log 3 + \log 2}}$

49. $e^{2x} - 3e^x = 4$
Let $u = e^x$ so that $u^2 = e^{2x}$.
$u^2 - 3u - 4 = 0$
$(u-4)(u+1) = 0$
$u = -1$ or 4
$e^x = -1$ or 4.
-1 is not in the range of e^x so we proceed with $e^x = 4$.
$\ln e^x = \ln 4$
$x = \ln 4$

53. Use $A = Pe^{it}$ with $P = 850$, $i = 0.0725$, $t = 2.25$. $A = 850e^{(0.0725)(2.25)} \approx 850(1.1771838) \approx \$1000.61$

57. $A = Pe^{it}$, $A = 5000$, $i = 0.1$, $t = 6$

$5000 = Pe^{(0.1)(6)}$; $P = \dfrac{5000}{e^{0.6}} \approx$ $\$2744.06$

61. $A = Pe^{it}$, $A = 3P$, $i = 0.05$; find t.

$3P = Pe^{0.05t}$

$3 = e^{0.05t}$

$\ln 3 = 0.05t$ $\qquad \ln e^x = x.$

$t = \dfrac{\ln 3}{0.05} = 20 \ln 3 \approx 21.97$ years.

65. $q = q_0e^{-0.000124t}$, $q = 0.3q_0$

$0.3q_0 = q_0e^{-0.000124t}$

$0.3 = e^{-0.000124t}$

$\ln 0.3 = \ln e^{-0.000124t}$

$\ln 0.3 = -0.000124t$

$t = \dfrac{\ln 0.3}{-0.000124} \approx 9709$. Thus the charcoal is about 10,000 years old.

69. $\alpha = 10 \log \frac{I}{I_0}$, $I = 20I_o$

so $\alpha = 10 \log \frac{20I_0}{I_0}$

$\alpha = 10 \log 20 \approx 13$.

73. $q = 1 - e^{-t}$, $t = 3$

$q = 1 - e^{-3} \approx 0.95 \approx 95\%$

77. $q = 1 - e^{-t}$

$e^{-t} = 1 - q$

$\ln e^{-t} = \ln(1 - q)$

$-t = \ln(1 - q)$

$t = -\ln(1 - q)$

81. $y = \dfrac{e^{-x^2/2}}{\sqrt{2\pi}}$

$y\sqrt{2\pi} = e^{-x^2/2}$

$\ln y\sqrt{2\pi} = \ln e^{-x^2/2}$

$\ln y\sqrt{2\pi} = -\frac{x^2}{2}$

$x^2 = -2 \ln(y\sqrt{2\pi})$

$x = \pm\sqrt{-2\ln(y\sqrt{2\pi})}$.

85.

x	x^2	2^x
5	25	32
10	100	1,024
20	400	1,048,576
40	1,600	1.09951×10^{12}

89. $q = q_oe^{rt}$, $q = 1.15q_o$ (remember an increase of 15% puts the population at 115%), $t = 15$.

$1.15q_o = q_oe^{15r}$

$1.15 = e^{15r}$

$\ln 1.15 = \ln e^{15r}$

$\ln 1.15 = 15r$

$r = \dfrac{\ln 1.15}{15} \approx 0.009317$. Thus for this bacteria

$q = q_oe^{0.009317t}$. Now we need to find t for which

$q = 2q_o$.

$2q_o = q_oe^{0.009317t}$

$2 = e^{0.009317t}$

$\ln 2 = \ln e^{0.009317t}$

$\ln 2 = 0.009317t$

$t = \dfrac{\ln 2}{0.0093} \approx 74.4$ hours.

93. $A = 106$, $i = 0.12$, $n = 1$, $t = 5$: find P.

$A = P\left(1 + \dfrac{i}{n}\right)^{nt}$

$106 = P(1 + 0.12)^5$

$P = \dfrac{106}{1.12^5} \approx 60.1$ talents

Chapter 9 Review

1. If $b > 1$ the exponential function is increasing; if $0 < b < 1$ the function is decreasing.

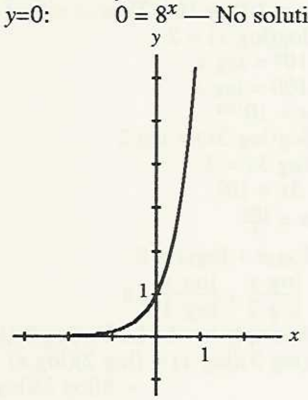
3. $\dfrac{4^{\sqrt{50}}}{4^{\sqrt{8}}} = 4^{5\sqrt{2}-2\sqrt{2}} = 4^{3\sqrt{2}}$

5. $f(x) = 8^x$; increasing

Intercepts:

x=0: $\quad y = 8^0 = 1$

y=0: $\quad 0 = 8^x$ — No solution

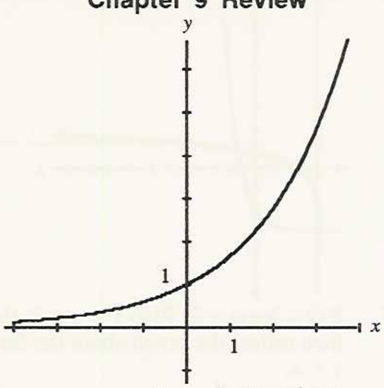


7. $f(x) = 0.6^{-x} = (\frac{3}{5})^{-x} = (\frac{5}{3})^x$; increasing

Intercepts:

x=0: $\quad y = (\frac{5}{3})^0 = 1$

y=0: $\quad 0 = (\frac{5}{3})^x$ – No solution

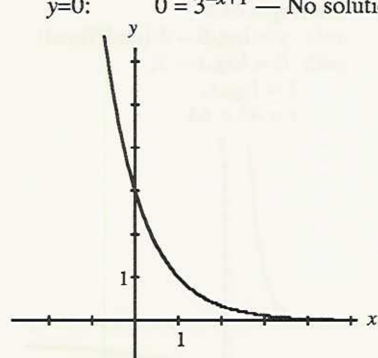


9. $f(x) = 3^{-x+1} = 3^{-(x-1)} = (\frac{1}{3})^{x-1}$; decreasing

Intercepts:

x=0: $\quad y = 3^1 = 3$

y=0: $\quad 0 = 3^{-x+1}$ — No solution



11. $9^x = 27$

$(3^2)^x = 3^3$

$3^{2x} = 3^3$

$2x = 3$

$x = \frac{3}{2}$

13. $10^{-2x} = 1000$

$10^{-2x} = 10^3$

$-2x = 3$

$x = -\frac{3}{2}$

15. $3^{x/2} = 9$

$3^{x/2} = 3^2$

$\frac{x}{2} = 2$

$x = 4$

17. $2^{-x} = 0.125$; $2^{-x} = \frac{1}{8}$

$2^{-x} = 2^{-3}$

$x = 3$

19. $\log_{10}0.001 = x$

$10^x = 10^{-3}$

$x = -3$

21. $5(3 \log_4 \frac{1}{8} + 2 \log_{10}100)$

$\log_4 \frac{1}{8} = x$

$4^x = \frac{1}{8}$

$2^{2x} = 2^{-3}$

$2x = -3$

$x = -\frac{3}{2}$

so

$5(3 \log_4 \frac{1}{8} + 2 \log_{10}100)$

$= 5[3(-\frac{3}{2}) + 2(2)] = -\frac{5}{2}$

23. $4^{-1} = 0.25$

25. $2^8 = y$

27. $m^{y+1} = x$

29. $5 = y^{2x}$, $\log_y 5 = 2x$

31. $\log_{x-1} 5 = y$

33. $\log_{5x} 3y = 2$

35. $\log_x 16 = 4$

$x^4 = 16$

$x^4 = 2^4$

$x = 2$

37. $\log_x \frac{1}{8} = 3$
$x^3 = \frac{1}{8}$
$x^3 = 2^{-3}$
$x^3 = (\frac{1}{2})^3$
$x = \frac{1}{2}$

39. $\log_4 100$
$4^3 = 64$
$4^4 = 256$
so $3 < \log_4 100 < 4$

41. m

43. $\log_2 3x = -3$
$2^{-3} = 3x$, $\frac{1}{8} = 3x$, $\frac{1}{24} = x$

45. $\log_3 \frac{5}{x} = \log_3(2x-3)$
$\frac{5}{x} = 2x - 3$
$5 = 2x^2 - 3x$
$2x^2 - 3x - 5 = 0$
$(2x-5)(x+1) = 0$
$x = \frac{5}{2}$ or -1. -1 does not check (since $\log_a x$ does not exist if $x \le 0$), so the solution is $\frac{5}{2}$.

47. $\log_2(-3x) = \log_2 \frac{1}{8}$
$-3x = \frac{1}{8}$
$x = -\frac{1}{24}$

49. $\log_3(x-2) = \log_2 \frac{1}{16}$
$\log_3(x-2) = -4$
$3^{-4} = x - 2$
$x = 2\frac{1}{81}$

51. $\log_2 5 + \log_2(3-2x) = \log_2 x$
$\log_2[5(3-2x)] = \log_2 x$
$5(3-2x) = x$
$15 = 11x$
$x = \frac{15}{11}$, which checks.

53. $\log_2(x-4) + \log_2(x+3)$
$= \log_2(x^2 - 3x + 2)$
$\log_2[(x-4)(x+3)] = \log_2(x^2 - 3x + 2)$
$x^2 - x - 12 = x^2 - 3x + 2$
$2x = 14$,
$x = 7$, which checks.

55. $\log_4 x^3 = \log_4 16^9$
$x^3 = 16^9$
$x^3 = (16^3)^3$
$x = 16^3 = 4{,}096$, which checks.

57. $\log_4 \frac{8yz^3}{x^2} = \log_4 8yz^3 - \log_4 x^2$
$= \log_4 8 + \log y + \log_4 z^3 - 2\log_4 z$
Let $m = \log_4 8$; then $4^m = 8$
$2^{2m} = 2^3$
$2m = 3$
$m = 3/2$
Then $\log_4 8 + \log_4 y + \log_4 z^3 - 2\log_4 z$
$= \frac{3}{2} + \log_4 y + 3\log_4 z - 2\log_4 z$.

59. $\log_a 30 = \log_a(2\cdot3\cdot5)$
$= \log_a 2 + \log_a 3 + \log_a 5$
$= 0.3562 + 0.5646 + 0.8271 = 1.7479$

61. $\log_a 0.2 = \log_a \frac{1}{5} = \log_a 1 - \log_a 5$
$= 0 - 0.8271 = -0.8271$

63. $\log 6{,}250 \approx 3.795880017 \approx 3.7959$

65. $\log 0.000\,000\,000\,000\,000\,004\,13$
$\approx \log(4.13 \times 10^{-18})$
$\approx \log 4.13 + \log 10^{-18}$
$\approx 0.61595 + (-18) \approx -17.3840$

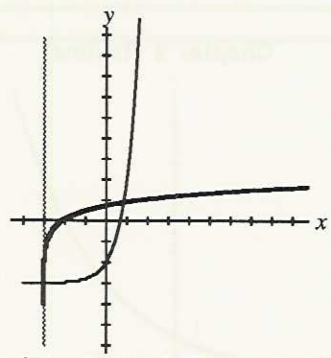
67. $\log_{20}1000 \approx \frac{\log 1000}{\log 20} \approx 2.305865361$
≈ 2.3059

69. $\log_{0.25}20 \approx \frac{\log 20}{\log 0.25} \approx -2.160964047$
≈ -2.1610

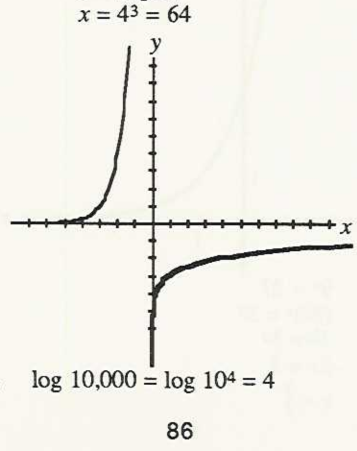
71. $\ln 0.0035 \approx -5.6550$

73. $10^{-2.5}$: 2.5 $\boxed{+/-}$ $\boxed{\text{shift}}$ $\boxed{\log}$ or
2.5 $\boxed{+/-}$ $\boxed{10^x}$ $\boxed{TI\text{-}81}$ 10 $\boxed{\wedge}$ $\boxed{(-)}$
2.5 $\boxed{\text{ENTER}}$
0.0032

75. $f(x) = \log_5(x+3)$; Find f^{-1}, graph that, then reflect the graph about the line $y = x$.
$y = \log_5(x+3)$
$x = \log_5(y+3)$
$5^x = y + 3$
$y = 5^x - 3$
$f^{-1}(x) = 5^x - 3$
Intercepts of f
$x=0$: $y = \log_5 3$
$y=0$: $0 = \log_5(x+3)$
$5^0 = x + 3$
$-2 = x$



77. $f(x) = \log_4 x - 3$; Find f^{-1}, graph that, then reflect the graph about the line $y = x$.
$y = \log_4 x - 3$
$x = \log_4 y - 3$
$x + 3 = \log_4 y$
$4^{x+3} = y$
$f^{-1}(x) = 4^{x+3}$
Intercepts of f
$x=0$: $y = \log_4 0 - 3$ (undefined)
$y=0$: $0 = \log_4 x - 3$,
$3 = \log_4 x$,
$x = 4^3 = 64$



79. $\log 10{,}000 = \log 10^4 = 4$

81. $\ln e^{\sqrt{3}} = \sqrt{3}$

83. $10^{\log 30} = 30$

85. $e^{\ln 2x} + \ln e^{2x} = 2x + 2x = 4x$

87. $A_0 = 2500$, $i = 0.0625$, $t = 3.5$: $A_0 = 2500e^{(0.0625)(3.5)} \approx 2500(1.244520108)$
$\approx 3{,}111.300269 \approx \$3{,}111.30$

89. $(\sqrt{8})^{x-2} = 8^{3x}$
$(8^{1/2})^{x-2} = 8^{3x}$
$8^{(x-2)/2} = 8^{3x}$
$\frac{x-2}{2} = 3x$
$x - 2 = 6x$
$-\frac{2}{5} = x$

91. $80 = 2^{x+3}$
$\log 80 = \log 2^{x+3}$
$\log 80 = (x+3)\log 2$
$\frac{\log 80}{\log 2} = x + 3$,
$x = \frac{\log 80}{\log 2} - 3 = 3.32$

93. $4.8 = 1.6^x$: $\log 4.8 = \log 1.6^x$
$\log 4.8 = x \log 1.6$
$x = \frac{\log 4.8}{\log 1.6} \approx 3.34$

95. $\log_5 30 = x$: $x = \log_5 30 = \frac{\log 30}{\log 5} \approx$
2.11

97. $\log_3 784 = 2x$
$2x = \frac{\log 784}{\log 3}$
$x = \frac{\log 784}{2\log 3} \approx 3.03$

99. $\log x^2 = (\log x)^2$
$2\log x = (\log x)^2$
let $u = \log x$, so we have
$2u = u^2$,
$u^2 - 2u = 0$
$u(u-2) = 0$
$u = 0$ or 2, so
$\log x = 0$ or $\log x = 2$
$x = 10^0$ or 10^2. Thus, $x = 1$ or 100.

101. $\log(\log x) = 2$
$10^2 = \log x$
$100 = \log x$
$x = 10^{100}$

103. $\log(\log 3x) = \log 2$
$\log 3x = 2$
$3x = 10^2$
$x = \frac{100}{3}$

105. $\log_2 x + \log_3 x = 8$
$\frac{\log x}{\log 2} + \frac{\log x}{\log 3} = 8$
Multiply by the LCD, $(\log 2)(\log 3)$.
$(\log 3)(\log x) + (\log 2)(\log x)$
$= 8(\log 3)(\log 2)$
$(\log x)(\log 3 + \log 2)$
$= 8(\log 3)(\log 2)$
$\log x = \frac{8(\log 3)(\log 2)}{\log 2 + \log 3}$
$x = 10^{\frac{8(\log 3)(\log 2)}{\log 2 + \log 3}}$

107. $3 = e^x - 4e^{-x}$
Multiply each term by e^x.
$3e^x = e^{2x} - 4e^0$,
$(e^x)^2 - 3e^x - 4 = 0$.
Let $u = e^x$ so
$u^2 - 3u - 4 = 0$
$(u-4)(u+1) = 0$
$u = -1$ or 4, so $e^x = -1$ or $e^x = 4$.
$e^x = -1$ is impossible since the range

of $f(x) = e^x$ is $y > 0$, so we solve
$e^x = 4$
$\ln e^x = \ln 4$
$x = \ln 4.$

109. $5^{x-1} = e^{x+1}$
$\ln 5^{x-1} = \ln e^{x+1}$
$(x-1)\ln 5 = x + 1$
$x \ln 5 - \ln 5 = x + 1$
$x \ln 5 - x = 1 + \ln 5$
$x(\ln 5 - 1) = \ln 5 + 1$
$x = \dfrac{\ln 5 + 1}{\ln 5 - 1}$

111. $\ln x = \dfrac{12}{\ln x - 4}$
$(\ln x)^2 - 4 \ln x = 12$
Replace $\ln x$ by u:
$u^2 - 4u - 12 = 0$,
$u = -2$ or 6
$\ln x = -2$ or $\ln x = 6$
$x = e^{-2}$ or e^6.

113. Using $A = Pe^{it}$, we want $A = 2P$ with $t = 9$, so
$2P = Pe^{9i}$
$2 = e^{9i}$
$\ln 2 = \ln e^{9i}$
$\ln 2 = 9i$
$i = \dfrac{\ln 2}{9} \approx 0.0770$ or 7.7%.

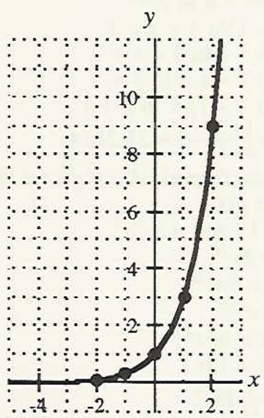
Chapter 9 Test

1. $\dfrac{8^{\sqrt{18}}}{8^{\sqrt{32}}} = 8^{\sqrt{18}-\sqrt{32}} = 8^{3\sqrt{2}-4\sqrt{2}}$
$= 8^{-\sqrt{2}} = \dfrac{1}{8^{\sqrt{2}}}$

3. $f(x) = 3^x$
Additional Points:

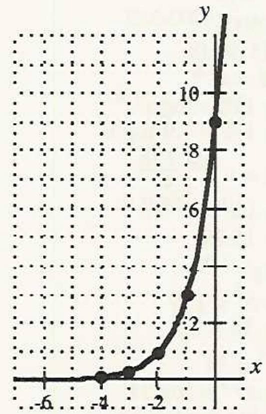
x	-2	-1	0	1	2
y	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

This is an increasing function since the base, 3, is greater than 1.



5. $f(x) = 3^{x+2}$
Increasing; the graph of $y = 3^x$ shifted two units to the left.
Additional Points:

x	-4	-3	-2	-1	0
y	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9



7. $9^x = 3$
$(3^2)^x = 3^1$
$3^{2x} = 3^1$
$2x = 1$
$x = \frac{1}{2}$

9. $25^x = \frac{1}{125}$
$(5^2)^x = 5^{-3}$
$5^{2x} = 5^{-3}$
$2x = -3$
$x = -\frac{3}{2}$

11. $100^{-x} = 0.001$
$(10^2)^{-x} = 10^{-3}$
$10^{-2x} = 10^{-3}$
$-2x = -3$
$x = \frac{3}{2}$

13. $3 \log_2 \frac{1}{16}$
$3 \log_2 2^{-4}$
$3(-4)$
-12

15. $\log_a x = y$ if and only if $a^y = x$, $a > 0$, $a \neq 1$.

17. $3^2 = x - 3$

Base and exponent go on opposite sides of the "=".

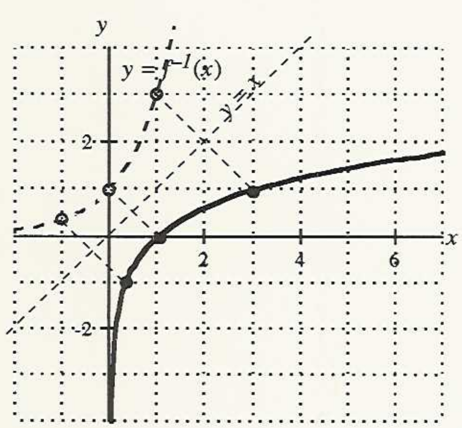
19. $2^5 = 32$ Base is 2; exponent is 5
$\log_2 32 = 5$

21. $z = y^{2x-1}$ Base is y; exponent is $2x - 1$
$\log_y z = 2x - 1$

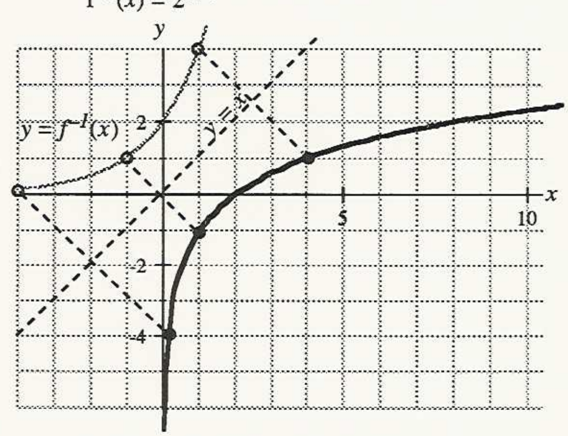
23. $\log_{10} (2x - 5) = 3$
$2x - 5 = 10^3$
$2x - 5 = 1000$
$2x = 1005$
$x = 502\frac{1}{2}$

Form the inverse function, graph it, and "flip" it about the line $y = x$.

25. $f(x) = \log_3 x$
$y = \log_3 x$
$x = \log_3 y$
$y = 3^x$
$f^{-1}(x) = 3^x$



27. $f(x) = \log_2 x - 1$
$y = \log_2 x - 1$
$x = \log_2 y - 1$
$x + 1 = \log_2 y$
$y = 2^{x+1}$
$f^{-1}(x) = 2^{x+1}$



29. $\log_2(x^2 - 14x) = \log_2 32$
$x^2 - 14x = 32$
$x^2 - 14x - 32 = 0$
$(x - 16)(x + 2) = 0$
$x = -2$ or 16

31. $\log_3(5x - 1) = -2$
$5x - 1 = 3^{-2}$
$5x = \frac{10}{9}$
$x = \frac{2}{9}$

33. $\log_2(x + 6) - \log_2(x - 1) = 5$
$\log_2 \frac{x + 6}{x - 1} = 5$
$\frac{x + 6}{x - 1} = 2^5$
$x + 6 = 32(x - 1)$
$x + 6 = 32x - 32$
$38 = 31x$
$\frac{38}{31} = x$

35. $2 \log_{10}(x + 2) = \log_{10}(x + 14)$
$\log_{10}(x + 2)^2 = \log_{10}(x + 14)$
$(x + 2)^2 = x + 14$
$x^2 + 3x - 10 = 0$
$(x + 5)(x - 2) = 0$
$x = -5$ or 2.
The value -5 is not a solution, since if $x = -5$ then $\log_{10}(x + 2) = \log_{10}(-3)$, which is not defined.
Thus the solution is 2.

37. $\log_3 \frac{9x^3y}{z^4} = \log_3 9x^3y - \log_3 z^4$
$= \log_3 9 + \log_3 x^3 + \log_3 y - 4 \log_3 z$
$= 2 + 3 \log_3 x + \log_3 y - 4 \log_3 z$

39. $\log_a 12 = \log_a(3 \cdot 2^2) = \log_a 3 + \log_a 2^2$
$= \log_a 3 + 2 \log_a 2 = 0.5646 +$
$2(0.3562) = 1.2770$

41. $\log_a 0.4 = \log_a \frac{2}{5} = \log_a 2 - \log_a 5$
$= 0.3562 - 0.8271 = -0.4709$

43. $\log 0.000\,000\,001\,03$
$\log(1.03 \times 10^{-9})$
$\log 1.03 + \log 10^{-9}$
$0.0128 - 9$
-8.9872

45. $\ln 1000$
6.9078

47. $3\sqrt{e}$
4.9462 3 $\boxed{\times}$ 1 $\boxed{\text{SHIFT}}$ $\boxed{\ln}$ $\boxed{\sqrt{x}}$ $\boxed{=}$
$\boxed{\text{TI-81}}$ 3 $\boxed{\times}$ $\boxed{\text{2nd}}$ $\boxed{x^2}$ $\boxed{(}$ $\boxed{\text{2nd}}$
$\boxed{\text{LN}}$ 1 $\boxed{)}$ $\boxed{\text{ENTER}}$

49. $[\ln e^x]^2$
$[x]^2$
x^2

51. $10^{\log 27}$
27

53. $A = A_0 e^{it}$
$= 2500e^{0.075 \cdot 3.25}$
$= \$3190.06$

55. $178 = x^{1.9}$
$\log 178 = \log x^{1.9}$
$\log 178 = 1.9 \log x$
$\log x = \frac{\log 178}{1.9}$
$x = 10^{(\log 178)/1.9}$
$x \approx 15.29$

57. $\log_5 x = 3$
$x = 5^3 = 125$

59. $\log_x 34 = 2.5$
$x^{2.5} = 34$
$\log x^{2.5} = \log 34$
$2.5 \log x = \log 34$
$\log x = \frac{\log 34}{2.5}$
$x = 10^{(\log 34)/2.5}$
$x \approx 4.10$

61. $4^{x-2} = 3^{6x-1}$
$\log 4^{x-2} = \log 3^{6x-1}$
$(x - 2)\log 4 = (6x - 1)\log 3$
$x \log 4 - 2 \log 4 = 6x \log 3 - \log 3$
$x \log 4 - 6x \log 3 = 2 \log 4 - \log 3$
$x(\log 4 - 6 \log 3) = 2 \log 4 - \log 3$
$x = \frac{2 \log 4 - \log 3}{\log 4 - 6 \log 3}$
$x \approx -0.3216$

63. $\log(\log x^2) = 1$
$\log x^2 = 10$
$x^2 = 10^{10}$
$x = \pm(10^{10})^{1/2}$
$x = \pm 10^5$

65. $\log_2 x + \log_3 x = 3$
$\frac{\log x}{\log 2} + \frac{\log x}{\log 3} = 3$
$\log x\left(\frac{1}{\log 2} + \frac{1}{\log 3}\right) = 3$
$\log x = 3\left(\frac{1}{\log 2} + \frac{1}{\log 3}\right)^{-1}$
$x = 10^{\left(3\left(\frac{1}{\log 2} + \frac{1}{\log 3}\right)^{-1}\right)}$
$x \approx 3.5787$
3 $\boxed{\times}$ $\boxed{(}$ 2 $\boxed{\log}$ $\boxed{1/x}$ $\boxed{+}$ 3 $\boxed{\log}$
$\boxed{1/x}$ $\boxed{)}$ $\boxed{\text{SHIFT}}$ $\boxed{\log}$
$\boxed{\text{TI-81}}$ $\boxed{\text{2nd}}$ $\boxed{\text{LOG}}$ $\boxed{(}$ 3 $\boxed{(}$ $\boxed{(}$
$\boxed{\text{LOG}}$ 2 $\boxed{)}$ $\boxed{x^{-1}}$ $\boxed{+}$ $\boxed{(}$ $\boxed{\text{LOG}}$
3 $\boxed{)}$ $\boxed{x^{-1}}$ $\boxed{)}$ $\boxed{x^{-1}}$ $\boxed{)}$ $\boxed{\text{ENTER}}$

67. $3^{x-1} = e^{x+1}$
$\ln 3^{x-1} = \ln e^{x+1}$
$(x - 1)\ln 3 = x + 1$
$x \ln 3 - \ln 3 = x + 1$
$x \ln 3 - x = \ln 3 + 1$
$x(\ln 3 - 1) = \ln 3 + 1$
$x = \frac{\ln 3 + 1}{\ln 3 - 1} \approx 21.2814$

69. In the formula $A = Pe^{it}$, we want to find i if
$t = 12$ and $A = 2P$.
$A = Pe^{it}$
$2P = Pe^{12i}$
$2 = e^{12i}$
$\ln 2 = \ln e^{12i}$
$\ln 2 = 12i$
$i = \frac{1}{12} \ln 2$
$i \approx 0.05776$, or 5.8%.


DISCOVER
CARD
Cashback Bonus

Exercise 10–1

1. [1] $-7 = x + 10y$
[2] $4 = -2x + 5y$
[3] $-10 = 25y$ $\quad\Leftarrow 2[1] + [2]$
$y = -\frac{2}{5}$
[4] $-3 = x$ Put value of y from [3] into [1].
$(-3, -\frac{2}{5})$

5. [1] $\frac{14}{3} = \frac{2}{3}x + \frac{3}{5}y$
[2] $-6 = x - \frac{2}{5}y$
Multiply [1] by 15 and multiply [2] by 5.
[1] $70 = 10x + 9y$
[2] $-30 = 5x - 2y$
[3] $130 = 13y$ $\quad\Leftarrow [1] + (-2)[2]$
$y = 10$
[4] $-10 = 5x$ Put value of y into [2].
$x = -2$
$(-2, 10)$

9. [1] $2 = 10x + 2y$
[2] $-5 = 10y$
$y = -\frac{1}{2}$
[3] $3 = 10x$ Put value of y into [1].
$x = \frac{3}{10}$
$(\frac{3}{10}, -\frac{1}{2})$

13. [1] $-10 = -2x - 3y$
[2] $35 = x - 6y$
[3] $60 = -15y$ $\quad\Leftarrow [1] + 2[2]$
$y = -4$
[4] $-55 = -5x$ $\quad\Leftarrow 2[1] - [2]$
$x = 11$
$(11, -4)$

17. [1] $12 = 4x + 4y$
[2] $18 = 9x + 9y$
Divide [1] by 4 and [2] by 9.
[1] $3 = x + y$
[2] $2 = x + y$
[3] $1 = 0$ $\quad\Leftarrow [1] - [2]$
No solution: *Inconsistent*

21. [1] $-63 = -8x + 8y$
[2] $\frac{17}{2} = x + 4y$
[3] $5 = 40y$ $\quad\Leftarrow [1] + 8[2]$
[3] $y = \frac{1}{8}$
[4] $-80 = -10x$ $\quad\Leftarrow [1] + (-2)[2]$
[4] $x = 8$
$(8, \frac{1}{8})$

25. [1] $-9 = x + y - 5z$
[2] $9 = -x + y + 2z$
[3] $-4 = 5x + 2y$
[4] $0 = 2y - 3z$ $\quad\Leftarrow [1] + [2]$
[5] $41 = 7y + 10z$ $\quad\Leftarrow 5[2] + [3]$
[6] $-82 = -41z$ $\quad\Leftarrow 7[4] + (-2)[5]$
$z = 2$
[7] $6 = 2y$ Insert value of z into [4].
$y = 3$
[8] $-2 = x$ Insert value of y and z into [1].
$(-2, 3, 2)$

29. [1] $29 = -x + 3y - 3z$
[2] $30 = x + \frac{4}{5}y - 5z$
[3] $-30 = -3x - 3y + \frac{12}{5}z$
Multiply [2] by 5 and [3] by 5.
[1] $29 = -x + 3y - 3z$
[2] $150 = 5x + 4y - 25z$
[3] $-150 = -15x - 15y + 12z$
Divide [3] by 3.
[1] $29 = -x + 3y - 3z$
[2] $150 = 5x + 4y - 25z$
[3] $-50 = -5x - 5y + 4z$
[4] $295 = 19y - 40z$ $\quad\Leftarrow 5[1] + [2]$
[5] $100 = -y - 21z$ $\quad\Leftarrow [2] + [3]$
[6] $2195 = -439z$ $\quad\Leftarrow [4] + 19[5]$
$z = -5$
[7] $-5 = -y$ Insert value of z into [5].
$y = 5$
[8] $-1 = -x$ Insert value of y and z into [1].
$x = 1$
$(1, 5, -5)$

33. [1] $0 = y + z$
[2] $0 = -3x + 2y + 2z$
[3] $-40 = 6x + 2y - 6z$
Divide [3] by 2.
[1] $0 = y + z$
[2] $0 = -3x + 2y + 2z$
[3] $-20 = 3x + y - 3z$
[4] $-20 = 3y - z$ $\quad\Leftarrow [2] + [3]$
[5] $-20 = 4y$ $\quad\Leftarrow [1] + [4]$
$y = -5$
[6] $5 = z$ Insert value of y into [1].
[7] $0 = 3x$ Insert value of y and z into [3].
$x = 0$
$(0, -5, 5)$

37. $\frac{L}{W} = \frac{8}{5}$
$5L = 8W$
$5L - 8W = 0$
$P = 2L + 2W$; $P = 44$
$44 = 2L + 2W$
$22 = L + W$
Thus we have the system of equations $\begin{matrix} 5L - 8W = 0 \\ L + W = 22 \end{matrix}$.
Solving this system gives $L = 13\frac{7}{13}$ cm, $W = 8\frac{6}{13}$ cm.

41. $W = \frac{1}{2}L - 10$
$P = 150 = 2L + 2W$
$75 = L + W$
Solve $\begin{matrix} L - 2W = 20 \\ L + W = 75 \end{matrix}$ to find $L = 56\frac{2}{3}$ mm, $W = 18\frac{1}{3}$ mm.

45. If the two investments are x and y then $x + y = 10{,}000$ $0.06x + 0.12y = 900$, or $x + 2y = 15{,}000$, so we solve the system $\begin{matrix} x + y = 10{,}000 \\ x + 2y = 15{,}000 \end{matrix}$ to find $x = y = \$5{,}000$.

49. Since the points satisify $y = mx + b$, we know $\begin{matrix} -1 = 5m + b \\ 6 = 8m + b \end{matrix}$, which we solve to find m and b: $m = \frac{7}{3}$, $b = -\frac{38}{3}$, so the equation is $y = \frac{7}{3}x - \frac{38}{3}$.

53. We plug the points $(-2, -3)$ and $(0, 2)$ into the equation $y = mx + b$, and obtain the system $\begin{matrix} -3 = -2m + b \\ 2 = 0m + b \end{matrix}$.
Solving this system gives $m = \frac{5}{2}$ and $b = 2$, so the equation is $y = \frac{5}{2}x + 2$.

Exercise 10–2

1. $\begin{bmatrix} 2 & \frac{2}{3} & -2 \\ -3 & 1 & 15 \end{bmatrix}$ Multiply [1] by 3, then divide [1] by 2.

$\begin{bmatrix} 3 & 1 & -3 \\ -3 & 1 & 15 \end{bmatrix}$ $[2] \Leftarrow [1] + [2]$

$\begin{bmatrix} 3 & 1 & -3 \\ 0 & 2 & 12 \end{bmatrix}$ Divide [2] by 2.

$\begin{bmatrix} 3 & 1 & -3 \\ 0 & 1 & 6 \end{bmatrix}$ $[1] \Leftarrow [2] - [1]$

$\begin{bmatrix} -3 & 0 & 9 \\ 0 & 1 & 6 \end{bmatrix}$ Set coefficients to 1.

$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 6 \end{bmatrix}$ Solution: $(-3, 6)$

5. $\begin{bmatrix} -3 & 4 & -1 \\ 4 & 1 & 14 \end{bmatrix}$ $[1] \Leftarrow 4[2] - [1]$

$\begin{bmatrix} 19 & 0 & 57 \\ 4 & 1 & 14 \end{bmatrix}$ Divide [1] by 19.

$\begin{bmatrix} 1 & 0 & 3 \\ 4 & 1 & 14 \end{bmatrix}$ $[2] \Leftarrow 4[1] - [2]$

$\begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & -2 \end{bmatrix}$ Rearrange rows and set coefficients to 1.

$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$ Solution: $(3, 2)$

7. $\begin{bmatrix} \frac{2}{5} & \frac{1}{3} & 1 \\ -2 & 2 & -16 \end{bmatrix}$ Multiply [1] by 15.

$\begin{bmatrix} 6 & 5 & 15 \\ -2 & 2 & -16 \end{bmatrix}$ Divide [2] by 2.

$\begin{bmatrix} 6 & 5 & 15 \\ -1 & 1 & -8 \end{bmatrix}$ $[1] \Leftarrow 6[2] + [1]$

$\begin{bmatrix} 0 & 11 & -33 \\ -1 & 1 & -8 \end{bmatrix}$ Divide [1] by 11.

$\begin{bmatrix} 0 & 1 & -3 \\ -1 & 1 & -8 \end{bmatrix}$ $[2] \Leftarrow [1] - [2]$

$\begin{bmatrix} 0 & 1 & -3 \\ 1 & 0 & 5 \end{bmatrix}$ Rearrange rows and set coefficients to 1.

$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -3 \end{bmatrix}$ Solution: $(5, -3)$

11. $\begin{bmatrix} \frac{1}{2} & 3 & 21 \\ 2 & -2 & 0 \end{bmatrix}$ Multiply [1] by 2.

$\begin{bmatrix} 1 & 6 & 42 \\ 2 & -2 & 0 \end{bmatrix}$ Divide [2] by 2.

$\begin{bmatrix} 1 & 6 & 42 \\ 1 & -1 & 0 \end{bmatrix}$ $[2] \Leftarrow [1] - [2]$

$\begin{bmatrix} 1 & 6 & 42 \\ 0 & 7 & 42 \end{bmatrix}$ Divide [2] by 7.

$\begin{bmatrix} 1 & 6 & 42 \\ 0 & 1 & 6 \end{bmatrix}$ $[1] \Leftarrow 6[2] - [1]$

$\begin{bmatrix} -1 & 0 & -6 \\ 0 & 1 & 6 \end{bmatrix}$ Rearrange rows and set coefficients to 1.

$\begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 6 \end{bmatrix}$ Solution: $(6, 6)$

15. $\begin{bmatrix} -2 & \frac{1}{3} & 14 \\ 4 & \frac{2}{3} & -20 \end{bmatrix}$ Multiply [1] by 3.

$\begin{bmatrix} -6 & 1 & 42 \\ 4 & \frac{2}{3} & -20 \end{bmatrix}$ Multiply [2] by 3.

$\begin{bmatrix} -6 & 1 & 42 \\ 12 & 2 & -60 \end{bmatrix}$ Divide [2] by 2.

$\begin{bmatrix} -6 & 1 & 42 \\ 6 & 1 & -30 \end{bmatrix}$ $[2] \Leftarrow [1] + [2]$

$\begin{bmatrix} -6 & 1 & 42 \\ 0 & 2 & 12 \end{bmatrix}$ Divide [2] by 2.

$\begin{bmatrix} -6 & 1 & 42 \\ 0 & 1 & 6 \end{bmatrix}$ $[1] \Leftarrow [2] - [1]$

$\begin{bmatrix} 6 & 0 & -36 \\ 0 & 1 & 6 \end{bmatrix}$ Rearrange rows and set coefficients to 1.

$\begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 6 \end{bmatrix}$ Solution: $(-6, 6)$

19. $\begin{bmatrix} -3 & 1 & -4 \\ 6 & 2 & 20 \end{bmatrix}$ Divide [2] by 2.

$\begin{bmatrix} -3 & 1 & -4 \\ 3 & 1 & 10 \end{bmatrix}$ $[2] \Leftarrow [1] + [2]$

$\begin{bmatrix} -3 & 1 & -4 \\ 0 & 2 & 6 \end{bmatrix}$ Divide [2] by 2.

$\begin{bmatrix} -3 & 1 & -4 \\ 0 & 1 & 3 \end{bmatrix}$ $[1] \Leftarrow [2] - [1]$

$\begin{bmatrix} 3 & 0 & 7 \\ 0 & 1 & 3 \end{bmatrix}$ Rearrange rows and set coefficients to 1.

$\begin{bmatrix} 1 & 0 & \frac{7}{3} \\ 0 & 1 & 3 \end{bmatrix}$ Solution: $(\frac{7}{3}, 3)$

23. $\begin{bmatrix} 1 & 1 & -5 & -9 \\ -1 & 1 & 2 & 9 \\ 5 & 2 & 0 & -4 \end{bmatrix}$ $[2] \Leftarrow [1] + [2]$, $[3] \Leftarrow 5[1] - [3]$

$\begin{bmatrix} 1 & 1 & -5 & -9 \\ 0 & 2 & -3 & 0 \\ 0 & 3 & -25 & -41 \end{bmatrix}$ $[1] \Leftarrow [2] - 2[1]$, $[3] \Leftarrow 3[2] - 2[3]$

$\begin{bmatrix} -2 & 0 & 7 & 18 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 41 & 82 \end{bmatrix}$ Divide [3] by 41.

$\begin{bmatrix} -2 & 0 & 7 & 18 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ $[1] \Leftarrow 7[3] - [1]$, $[2] \Leftarrow 3[3] + [2]$

$\begin{bmatrix} 2 & 0 & 0 & -4 \\ 0 & 2 & 0 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ Rearrange rows and set coefficients to 1.

$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ Solution: $(-2, 3, 2)$

27. $\begin{bmatrix} -1 & 3 & -3 & 15 \\ 1 & 4 & -3 & 22 \\ -3 & -3 & 6 & -20 \end{bmatrix}$ $[2] \Leftarrow [1] + [2]$, $[3] \Leftarrow -3[1] + [3]$

$\begin{bmatrix} -1 & 3 & -3 & 15 \\ 0 & 7 & -6 & 37 \\ 0 & -12 & 15 & -65 \end{bmatrix}$ $[1] \Leftarrow -[2] + 2[1]$, $[3] \Leftarrow 5[2] + 2[3]$

$\begin{bmatrix} -2 & -1 & 0 & -7 \\ 0 & 7 & -6 & 37 \\ 0 & 11 & 0 & 55 \end{bmatrix}$ Divide [3] by 11.

$\begin{bmatrix} -2 & -1 & 0 & -7 \\ 0 & 7 & -6 & 37 \\ 0 & 1 & 0 & 5 \end{bmatrix}$ $[1] \Leftarrow [3] + [1]$, $[2] \Leftarrow 7[3] - [2]$

$\begin{bmatrix} -2 & 0 & 0 & -2 \\ 0 & 0 & 6 & -2 \\ 0 & 1 & 0 & 5 \end{bmatrix}$ Rearrange rows and set coefficients to 1.

$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -\frac{1}{3} \end{bmatrix}$ Solution: $(1, 5, -\frac{1}{3})$

31. $\begin{bmatrix} 0 & 1 & 1 & 0 \\ -3 & 2 & 2 & 0 \\ 5 & 2 & -6 & -40 \end{bmatrix}$ $[2] \Leftarrow 2[1] + -1[2]$, $[3] \Leftarrow 2[1] + -1[3]$

$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 3 & 0 & 0 & 0 \\ -5 & 0 & 8 & 40 \end{bmatrix}$ $[3] \Leftarrow 5[2] + 3[3]$

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 24 & 120 \end{bmatrix}$$
Divide [3] by 24.

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$
[1] ⇐ 1[3] + −1[1]

$$\begin{bmatrix} 0 & -1 & 0 & 5 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$
Rearrange rows and set coefficients to 1.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$
Solution: (0, −5, 5)

35.
$$\begin{bmatrix} 1 & 0 & -1 & 0 & 5 \\ -2 & 4 & 0 & 2 & 4 \\ -2 & -2 & -3 & -2 & -6 \\ 2 & 0 & -5 & 5 & 21 \end{bmatrix}$$
[2] ⇐ 2[3] + [2]

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 5 \\ -6 & 0 & -6 & -2 & -8 \\ -2 & -2 & -3 & -2 & -6 \\ 2 & 0 & -5 & 5 & 21 \end{bmatrix}$$
[2] ⇐ 6[1] + [2]
[3] ⇐ 2[1] + [3]
[4] ⇐ 2[1] − [4]

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 5 \\ 0 & 0 & -12 & -2 & 22 \\ 0 & -2 & -5 & -2 & 4 \\ 0 & 0 & 3 & -5 & -11 \end{bmatrix}$$
Divide [2] by 2.

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 5 \\ 0 & 0 & -6 & -1 & 11 \\ 0 & -2 & -5 & -2 & 4 \\ 0 & 0 & 3 & -5 & -11 \end{bmatrix}$$
[3] ⇐ −2[2] + [3]
[4] ⇐ −5[2] + [4]

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 5 \\ 0 & 0 & -6 & -1 & 11 \\ 0 & -2 & 7 & 0 & -18 \\ 0 & 0 & 33 & 0 & -66 \end{bmatrix}$$
Divide [4] by 33.

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 5 \\ 0 & 0 & -6 & -1 & 11 \\ 0 & -2 & 7 & 0 & -18 \\ 0 & 0 & 1 & 0 & -2 \end{bmatrix}$$
[1] ⇐ [4] + [1]
[2] ⇐ 6[4] + [2]
[3] ⇐ 7[4] − [3]

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -2 \end{bmatrix}$$
Rearrange rows and set coefficients to 1.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$
Solution: (3, 2, −2, 1)

39.
$$\begin{bmatrix} 1 & \frac{1}{2} & 3 & -3 & -5 \\ 2 & -\frac{3}{2} & 3 & 5 & 16 \\ -6 & 0 & 0 & -1 & -8 \\ 1 & 0 & -6 & 0 & -1 \end{bmatrix}$$
Multiply [1] by 2, and [2] by 2.

$$\begin{bmatrix} 2 & 1 & 6 & -6 & -10 \\ 4 & -3 & 6 & 10 & 32 \\ -6 & 0 & 0 & -1 & -8 \\ 1 & 0 & -6 & 0 & -1 \end{bmatrix}$$
[2] ⇐ 3[1] + [2]

$$\begin{bmatrix} 2 & 1 & 6 & -6 & -10 \\ 10 & 0 & 24 & -8 & 2 \\ -6 & 0 & 0 & -1 & -8 \\ 1 & 0 & -6 & 0 & -1 \end{bmatrix}$$
Divide [2] by 2.

$$\begin{bmatrix} 2 & 1 & 6 & -6 & -10 \\ 5 & 0 & 12 & -4 & 1 \\ -6 & 0 & 0 & -1 & -8 \\ 1 & 0 & -6 & 0 & -1 \end{bmatrix}$$
[1] ⇐ 2[4] − [1]
[2] ⇐ 5[4] − [2]
[3] ⇐ 6[4] + [3]

$$\begin{bmatrix} 0 & -1 & -18 & 6 & 8 \\ 0 & 0 & -42 & 4 & -6 \\ 0 & 0 & -36 & -1 & -14 \\ 1 & 0 & -6 & 0 & -1 \end{bmatrix}$$
Divide [2] by 2.

$$\begin{bmatrix} 0 & -1 & -18 & 6 & 8 \\ 0 & 0 & -21 & 2 & -3 \\ 0 & 0 & -36 & -1 & -14 \\ 1 & 0 & -6 & 0 & -1 \end{bmatrix}$$
[1] ⇐ 6[3] + [1]
[2] ⇐ 2[3] + [2]

$$\begin{bmatrix} 0 & -1 & -234 & 0 & -76 \\ 0 & 0 & -93 & 0 & -31 \\ 0 & 0 & -36 & -1 & -14 \\ 1 & 0 & -6 & 0 & -1 \end{bmatrix}$$
Divide [2] by 31.

$$\begin{bmatrix} 0 & -1 & -234 & 0 & -76 \\ 0 & 0 & -3 & 0 & -1 \\ 0 & 0 & -36 & -1 & -14 \\ 1 & 0 & -6 & 0 & -1 \end{bmatrix}$$
[1] ⇐ −78[2] + [1]
[3] ⇐ −12[2] + [3]
[4] ⇐ −2[2] + [4]

$$\begin{bmatrix} 0 & -1 & 0 & 0 & 2 \\ 0 & 0 & -3 & 0 & -1 \\ 0 & 0 & 0 & -1 & -2 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$
Rearrange rows and set coefficients to 1.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$
Solution: $(1, -2, \frac{1}{3}, 2)$

43.
$$\begin{bmatrix} -3 & 6 & 5 & -3 & 1 \\ 4 & 0 & 2 & -3 & 36 \\ 2 & -5 & -3 & 0 & 9 \\ 1 & 2 & 5 & -4 & 29 \end{bmatrix}$$
[1] ⇐ 3[4] + [1]
[2] ⇐ 4[4] − [2]
[3] ⇐ 2[4] − [3]

$$\begin{bmatrix} 0 & 12 & 20 & -15 & 88 \\ 0 & 8 & 18 & -13 & 80 \\ 0 & 9 & 13 & -8 & 49 \\ 1 & 2 & 5 & -4 & 29 \end{bmatrix}$$
[1] ⇐ 3[2] − 2[1]
[3] ⇐ 9[2] − 8[3]
[4] ⇐ [2] − 4[4]

$$\begin{bmatrix} 0 & 0 & 14 & -9 & 64 \\ 0 & 8 & 18 & -13 & 80 \\ 0 & 0 & 58 & -53 & 328 \\ -4 & 0 & -2 & 3 & -36 \end{bmatrix}$$
[2] ⇐ −13[1] + 9[2]
[3] ⇐ −53[1] + 9[3]
[4] ⇐ [1] + 3[4]

$$\begin{bmatrix} 0 & 0 & 14 & -9 & 64 \\ 0 & 72 & -20 & 0 & -112 \\ 0 & 0 & -220 & 0 & -440 \\ -12 & 0 & 8 & 0 & -44 \end{bmatrix}$$
Divide [3] by −220.

$$\begin{bmatrix} 0 & 0 & 14 & -9 & 64 \\ 0 & 72 & -20 & 0 & -112 \\ 0 & 0 & 1 & 0 & 2 \\ -12 & 0 & 8 & 0 & -44 \end{bmatrix}$$
[1] ⇐ 14[3] − [1]
[2] ⇐ 20[3] + [2]
[4] ⇐ 8[3] − [4]

$$\begin{bmatrix} 0 & 0 & 0 & 9 & -36 \\ 0 & 72 & 0 & 0 & -72 \\ 0 & 0 & 1 & 0 & 2 \\ 12 & 0 & 0 & 0 & 60 \end{bmatrix}$$
Rearrange rows and set coefficients to 1.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix}$$
Solution: (5, −1, 2, −4)

47.
$$\begin{bmatrix} 60 & -10 & 116 \\ 10 & -30 & 8 \end{bmatrix}$$
Divide [1] and [2] by 2.

$$\begin{bmatrix} 30 & -5 & 58 \\ 5 & -15 & 4 \end{bmatrix}$$
[2] ⇐ −3[1] + 1[2]

$$\begin{bmatrix} 30 & -5 & 58 \\ -85 & 0 & -170 \end{bmatrix}$$
Divide [2] by 85.

$$\begin{bmatrix} 30 & -5 & 58 \\ -1 & 0 & -2 \end{bmatrix}$$
[1] ⇐ 30[2] + 1[1]

$$\begin{bmatrix} 0 & -5 & -2 \\ -1 & 0 & -2 \end{bmatrix}$$
Rearrange rows and set coefficients to 1.

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & \frac{2}{5} \end{bmatrix}$$
Solution: $i_1 = 2$, $i_2 = \frac{2}{5}$

51. Let t = required amount of 20% solution, f = required amount of 50% solution. Then $t + f = 500$. Now, 30% of the 500 liters is to be alcohol, or 150 liters. This alcohol comes from 20% of t and 50% of f, so that we also have the equation $0.20t + 0.50f = 150$, or $2t + 5f = 1500$. Thus we solve

$t + f = 500$
$2t + 5f = 1500$ for t and f. The solution is $(\frac{1000}{3}, \frac{500}{3})$, so we need $333\frac{1}{3}$ liters of the 20% solution and $166\frac{2}{3}$ liters of the 50% solution.

55. $\begin{bmatrix} 3 & 2 & 1 & 39 \\ 2 & 3 & 1 & 34 \\ 1 & 2 & 3 & 26 \end{bmatrix}$ $[1] \Leftarrow 3[3] + -1[1]$, $[2] \Leftarrow 2[3] + -1[2]$

$\begin{bmatrix} 0 & 4 & 8 & 39 \\ 0 & 1 & 5 & 18 \\ 1 & 2 & 3 & 26 \end{bmatrix}$ $[1] \Leftarrow 4[2] + -1[1]$, $[3] \Leftarrow 2[2] + -1[3]$

$\begin{bmatrix} 0 & 0 & 12 & 33 \\ 0 & 1 & 5 & 18 \\ -1 & 0 & 7 & 10 \end{bmatrix}$ Divide [1] by 3.

$\begin{bmatrix} 0 & 0 & 4 & 11 \\ 0 & 1 & 5 & 18 \\ -1 & 0 & 7 & 10 \end{bmatrix}$ $[2] \Leftarrow 5[1] + -4[2]$, $[3] \Leftarrow 7[1] + -4[3]$

$\begin{bmatrix} 0 & 0 & 4 & 11 \\ 0 & -4 & 0 & -17 \\ 4 & 0 & 0 & 37 \end{bmatrix}$ Rearrange rows and set coefficients to 1.

$\begin{bmatrix} 1 & 0 & 0 & \frac{37}{4} \\ 0 & 1 & 0 & \frac{17}{4} \\ 0 & 0 & 1 & \frac{11}{4} \end{bmatrix}$ Solution: $(\frac{37}{4}, \frac{17}{4}, \frac{11}{4})$

Exercises 10–3

1. $\begin{vmatrix} 1 & -4 \\ -3 & 3 \end{vmatrix} = 1(3) - (-3)(-4) = -9$

5. $\begin{vmatrix} -3\pi & -4\pi \\ 2 & 3 \end{vmatrix} = -3\pi(3) - 2(-4\pi) = -9\pi + 8\pi = -\pi$

9. $\begin{vmatrix} 2 & \frac{2}{3} & -1 \\ 4 & -1 & \frac{1}{2} \\ \mathbf{-3} & \mathbf{0} & \mathbf{-2} \end{vmatrix} = -3\begin{vmatrix} \frac{2}{3} & -1 \\ -1 & \frac{1}{2} \end{vmatrix} + (-2)\begin{vmatrix} 2 & \frac{2}{3} \\ 4 & -1 \end{vmatrix}$

$= -3(-\frac{2}{3}) - 2(-\frac{14}{3}) = \frac{34}{3}$

13. $\begin{vmatrix} -1 & 1 & 3 \\ 1 & -2 & 5 \\ \mathbf{0} & \mathbf{0} & \mathbf{7} \end{vmatrix} = 7\begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} = 7(1) = 7$

17. $\begin{vmatrix} \mathbf{\sqrt{2}} & \mathbf{0} & \mathbf{3} \\ \sqrt{8} & -5 & 2 \\ -\sqrt{2} & -1 & 7 \end{vmatrix} = \sqrt{2}\begin{vmatrix} -5 & 2 \\ -1 & 7 \end{vmatrix} + 3\begin{vmatrix} \sqrt{8} & -5 \\ -\sqrt{2} & -1 \end{vmatrix}$

$= \sqrt{2}(-33) + 3(-2\sqrt{2} - 5\sqrt{2})$
$= -33\sqrt{2} - 21\sqrt{2} = -54\sqrt{2}$

21. $\begin{vmatrix} 0 & -3 & 2 & \mathbf{0} \\ -2 & 5 & 10 & \mathbf{0} \\ -4 & 3 & 0 & \mathbf{1} \\ 2 & -3 & 1 & \mathbf{0} \end{vmatrix} = -\begin{vmatrix} \mathbf{0} & \mathbf{-3} & \mathbf{2} \\ -2 & 5 & 10 \\ 2 & -3 & 1 \end{vmatrix}$

$= -\left\{-(-3)\begin{vmatrix} -2 & 10 \\ 2 & 1 \end{vmatrix} + 2\begin{vmatrix} -2 & 5 \\ 2 & -3 \end{vmatrix}\right\}$
$= -[3(-22) + 2(-4)] = 74$

25. $\begin{vmatrix} 0 & 2 & -4 & 0 \\ -2 & 5 & 6 & 0 \\ 0 & 3 & 0 & 5 \\ 2 & -3 & 1 & 0 \end{vmatrix} = -5\begin{vmatrix} 0 & 2 & -4 \\ -2 & 5 & 6 \\ 2 & -3 & 1 \end{vmatrix}$

$= -5\left\{-2\begin{vmatrix} -2 & 6 \\ 2 & 1 \end{vmatrix} - 4\begin{vmatrix} -2 & 5 \\ 2 & -3 \end{vmatrix}\right\}$
$= -5[-2(-14) - 4(-4)] = -220$

29. $D_x = \begin{vmatrix} 0 & -4 \\ 4 & 9 \end{vmatrix} = 16$; $D_y = \begin{vmatrix} -3 & 0 \\ -1 & 4 \end{vmatrix} = -12$

$D = \begin{vmatrix} -3 & -4 \\ -1 & 9 \end{vmatrix} = -31$; $x = -\frac{16}{31}$, $y = \frac{12}{31}$

33. $D_x = \begin{vmatrix} 9 & -5 \\ -4 & -8 \end{vmatrix} = -92$; $D_y = \begin{vmatrix} -2 & 9 \\ -7 & -4 \end{vmatrix} = 71$

$D = \begin{vmatrix} -2 & -5 \\ -7 & -8 \end{vmatrix} = -19$; $x = \frac{92}{19}$, $y = -\frac{71}{19}$

37. $D_x = \begin{vmatrix} -6 & -3 & -3 \\ 7 & -4 & -6 \\ -3 & 2 & 0 \end{vmatrix} = -132$; $D_y = \begin{vmatrix} 9 & -6 & -3 \\ 1 & 7 & -6 \\ 3 & -3 & 0 \end{vmatrix} = 18$

$D_z = \begin{vmatrix} 9 & -3 & -6 \\ 1 & -4 & 7 \\ 3 & 2 & -3 \end{vmatrix} = -174$; $D = \begin{vmatrix} 9 & -3 & -3 \\ 1 & -4 & -6 \\ 3 & 2 & 0 \end{vmatrix} = 120$

$x = -\frac{11}{10}, y = \frac{3}{20}, z = -\frac{29}{20}$

41. $D_x = \begin{vmatrix} 4 & -1 & 0 \\ -3 & 9 & -3 \\ 7 & -1 & 0 \end{vmatrix} = 9$; $D_y = \begin{vmatrix} 1 & 4 & 0 \\ -4 & -3 & -3 \\ -3 & 7 & 0 \end{vmatrix} = 57$

$D_z = \begin{vmatrix} 1 & -1 & 4 \\ -4 & 9 & -3 \\ -3 & -1 & 7 \end{vmatrix} = 147$; $D = \begin{vmatrix} 1 & -1 & 0 \\ -4 & 9 & -3 \\ -3 & -1 & 0 \end{vmatrix} = -12$

$x = -\frac{3}{4}, y = -\frac{19}{4}, z = -\frac{49}{4}$

45. $D_x = \begin{vmatrix} 2 & 2 & 3 & -2 \\ -3 & 2 & 5 & -1 \\ -2 & 4 & -2 & -4 \\ 2 & 4 & 0 & -4 \end{vmatrix} = -40$; $D_y = \begin{vmatrix} 2 & 2 & 3 & -2 \\ -1 & -3 & 5 & -1 \\ -4 & -2 & -2 & -4 \\ -4 & 2 & 0 & -4 \end{vmatrix} = -432$

$D_z = \begin{vmatrix} 2 & 2 & 2 & -2 \\ -1 & 2 & -3 & -1 \\ -4 & 4 & -2 & -4 \\ -4 & 4 & 2 & -4 \end{vmatrix} = 64$; $D_w = \begin{vmatrix} 2 & 2 & 3 & 2 \\ -1 & 2 & 5 & -3 \\ -4 & 4 & -2 & -2 \\ -4 & 4 & 0 & 2 \end{vmatrix} = -408$

$D = \begin{vmatrix} 2 & 2 & 3 & -2 \\ -1 & 2 & 5 & -1 \\ -4 & 4 & -2 & -4 \\ -4 & 4 & 0 & -4 \end{vmatrix} = 32$; $x = -\frac{5}{4}, y = -\frac{27}{2}, z = 2, w = -\frac{51}{4}$

49. $D_x = \begin{vmatrix} 1 & -1 & 6 & -4 \\ 4 & -1 & 2 & 4 \\ 5 & -3 & 4 & 0 \\ 5 & -3 & 4 & -4 \end{vmatrix} = 136$; $D_y = \begin{vmatrix} 2 & 1 & 6 & -4 \\ 5 & 4 & 2 & 4 \\ 2 & 5 & 4 & 0 \\ 0 & 5 & 4 & -4 \end{vmatrix} = -232$

$D_z = \begin{vmatrix} 2 & -1 & 1 & -4 \\ 5 & -1 & 4 & 4 \\ 2 & -3 & 5 & 0 \\ 0 & -3 & 5 & -4 \end{vmatrix} = -112$; $D_w = \begin{vmatrix} 2 & -1 & 6 & 1 \\ 5 & -1 & 2 & 4 \\ 2 & -3 & 4 & 5 \\ 0 & -3 & 4 & 5 \end{vmatrix} = -68$

$D = \begin{vmatrix} 2 & -1 & 6 & -4 \\ 5 & -1 & 2 & 4 \\ 2 & -3 & 4 & 0 \\ 0 & -3 & 4 & -4 \end{vmatrix} = 104$; $x = \frac{17}{13}, y = -\frac{29}{13}, z = -\frac{14}{13}, w = -\frac{17}{26}$

53. (0, 0.5), (1, 1.8), (2, 3.0), (4, 5.0), (6, 7.6)
$X = 0 + 1 + 2 + 4 + 6 = 13$
$Y = 0.5 + 1.8 + 3 + 5 + 7.6 = 17.9$
$P = 0 \cdot 0.5 + 1 \cdot 1.8 + 2 \cdot 3 + 4 \cdot 5 + 6 \cdot 7.6 = 73.4$
$S = 0^2 + 1^2 + 2^2 + 4^2 + 6^2 = 57$
$N = 5$

Solve $\begin{matrix} 13m + 5b = 17.9 \\ 57m + 13b = 73.4 \end{matrix}$: $D_m = \begin{vmatrix} 17.9 & 5 \\ 73.4 & 13 \end{vmatrix} = -134.3$,

$D_b = \begin{vmatrix} 13 & 17.9 \\ 57 & 73.4 \end{vmatrix} = -66.1$, $D = \begin{vmatrix} 13 & 5 \\ 57 & 13 \end{vmatrix} = -116$,

$m = 1.16$, $b = 0.57$,
so $y = mx + b$ is $y = 1.16x + 0.57$.

57. $X = 1 + 2 + 3 + 4 = 10$
$Y = 3 + 5 + 6 + 9 = 23$
$P = 1\cdot3 + 2\cdot5 + 3\cdot6 + 4\cdot9 = 67$
$S = 1^2 + 2^2 + 3^2 + 4^2 = 30$
$N = 4$
Solve $\begin{matrix} 10m + 4b = 23 \\ 30m + 10b = 67 \end{matrix}$. $D_m = \begin{vmatrix} 23 & 4 \\ 67 & 10 \end{vmatrix} = -38$,
$D_b = \begin{vmatrix} 10 & 23 \\ 30 & 67 \end{vmatrix} = -20$, $D = \begin{vmatrix} 10 & 4 \\ 30 & 10 \end{vmatrix} = -20$, $m = \frac{19}{10}$
$b = 1$. Thus the line is $y = 1.9x + 1$. For the fifth year the line predicts $y = 1.9(5) + 1 = 10.5\%$ failures, and for the sixth year it predicts $y = 1.9(6) + 1 = 12.4\%$ failures.

61. Use area $= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ with $(x_1, y_1) = (-2, 6)$,
$(x_2, y_2) = (3, -2)$ and $(x_3, y_3) = (6,12)$:
area $= \frac{1}{2}\begin{vmatrix} -2 & 6 & 1 \\ 3 & -2 & 1 \\ 6 & 12 & 1 \end{vmatrix} =$
$\frac{1}{2}\left\{-2\begin{vmatrix} -2 & 1 \\ 12 & 1 \end{vmatrix} + (-6)\begin{vmatrix} 3 & 1 \\ 6 & 1 \end{vmatrix} + 1\begin{vmatrix} 3 & -2 \\ 6 & 12 \end{vmatrix}\right\}$
$= \frac{1}{2}[-2(-14) - 6(-3) + (48)] = 47$.

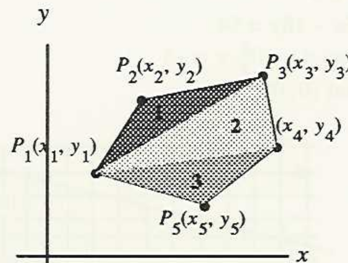
65. We use $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$ with $(x_1, y_1) = (2, -3)$ and $(x_2, y_2) = (5, 4)$.
$\begin{vmatrix} x & y & 1 \\ 2 & -3 & 1 \\ 5 & 4 & 1 \end{vmatrix} = 0$, $x\begin{vmatrix} -3 & 1 \\ 4 & 1 \end{vmatrix} - y\begin{vmatrix} 2 & 1 \\ 5 & 1 \end{vmatrix} + \begin{vmatrix} 2 & -3 \\ 5 & 4 \end{vmatrix} = 0$,
$-7x + 3y + 23 = 0$.

69. $\boxed{\begin{matrix} (x, y) & y = mx + b \\ (5, -1) & -1 = 5m + b \\ (8, 6) & 6 = 8m + b \end{matrix}}$,so solve the system $\begin{matrix} -1 = 5m + b \\ 6 = 8m + b \end{matrix}$
for m and b: $D_m = \begin{vmatrix} -1 & 1 \\ 6 & 1 \end{vmatrix} = -7$, $D_b = \begin{vmatrix} 5 & -1 \\ 8 & 6 \end{vmatrix} = 38$, $D = \begin{vmatrix} 5 & 1 \\ 8 & 1 \end{vmatrix}$
$= -3$, $m = \frac{7}{3}$, $b = -\frac{38}{3}$, so the line is $y = \frac{7}{3}x - \frac{38}{3}$.

73. $D_{i_1} = \begin{vmatrix} 50 & 12 & 5 \\ -40 & -20 & -10 \\ 60 & 10 & 5 \end{vmatrix} = -800$, $D_{i_2} = \begin{vmatrix} 35 & 50 & 5 \\ 30 & -40 & -10 \\ 15 & 60 & 5 \end{vmatrix} = 11000$,
$D_{i_3} = \begin{vmatrix} 35 & 12 & 50 \\ 30 & -20 & -40 \\ 15 & 10 & 60 \end{vmatrix} = -26800$, $D = \begin{vmatrix} 35 & 12 & 5 \\ 30 & -20 & -10 \\ 15 & 10 & 5 \end{vmatrix} = -600$,
$i_1 = \frac{4}{3}$, $i_2 = -\frac{55}{3}$, $i_3 = \frac{134}{3}$

77. $D = \begin{vmatrix} 0.12658 & 0.25315 \\ 0.88606 & 1.77213 \end{vmatrix} = 1.01264 \times 10^{-5}$,
$Dx = \begin{vmatrix} 0.37973 & 0.25315 \\ 2.65819 & 1.77213 \end{vmatrix} = 1.01264 \times 10^{-5}$,
$Dy = \begin{vmatrix} 0.12658 & 0.37973 \\ 0.88606 & 2.65819 \end{vmatrix} = 1.01264 \times 10^{-5}$,
so that $x = \frac{D_x}{D} = 1$, and $y = \frac{D_y}{D} = 1$. Thus, (1, 1) is the correct solution, which can be verified by substitution into the system itself.

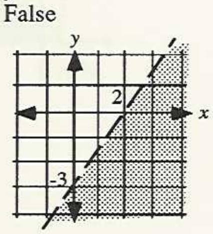
81. Consider the figures shown. They show how a polygon can be divided up into triangles. The area of the polygon is the sum of the areas of the triangles. Use the determinant of the previous problem to show that the following is a formula for the area of a four–sided polygon (a quadrilateral):
$\frac{1}{2}\{ (x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_4 - x_4y_3) + (x_4y_1 - x_1y_4) \}$.
Similarly, show that the area of a five–sided polygon is given by the formula:
$\frac{1}{2}\{ (x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_4 - x_4y_3) + (x_4y_5 - x_5y_4) + (x_5y_1 - x_1y_5) \}$.



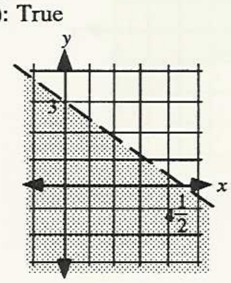
The area of the five–sided polygon is the sum of the areas marked 1, 2, and 3 in the figure. Each of these is a triangle.
$\text{Area}_{\text{total}} = \text{Area}_1 + \text{Area}_2 + \text{Area}_3$
$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_4 & y_4 & 1 \\ x_5 & y_5 & 1 \end{vmatrix}$$
$$= \frac{1}{2}[(x_2y_3 - x_3y_2) - (x_1y_3 - x_3y_1) + (x_1y_2 - x_2y_1)] +$$
$$\frac{1}{2}[(x_3y_4 - x_4y_3) - (x_1y_4 - x_4y_1) + (x_1y_3 - x_3y_1)] +$$
$$\frac{1}{2}[(x_4y_5 - x_5y_4) - (x_1y_5 - x_5y_1) + (x_1y_4 - x_4y_1)]$$
The circled terms add to zero.
$$= \frac{1}{2}\{ (x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_4 - x_4y_3) + (x_4y_5 - x_5y_4) + (x_5y_1 - x_1y_5) \}.$$
The solution for the four–sided figure is similar.

Exercise 10–4

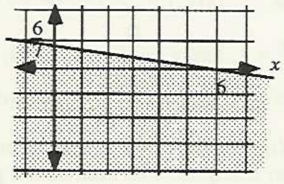
1. $3x - 2y > 6$
Graph $3x - 2y = 6$
Intercepts: $x = 2$, $y = -3$
Test Point (0, 0): False



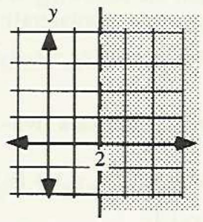
5. $9 - 2x > 3y$
Graph $2x + 3y = 9$
Intercepts: $x = 4\frac{1}{2}$, $y = 3$
Test Point (0, 0): True



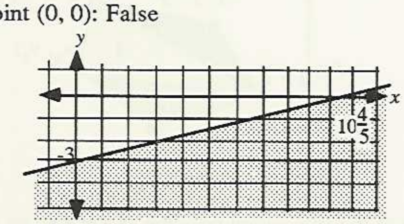
9. $6 \ge x + 7y$
Graph $x + 7y = 6$
Intercepts: $x = 6$, $y = \frac{6}{7}$
Test Point (0, 0): True



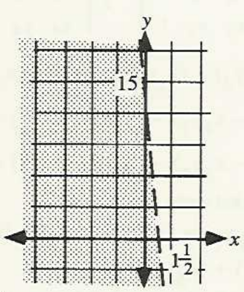
13. $x > 2$
Graph $x = 2$



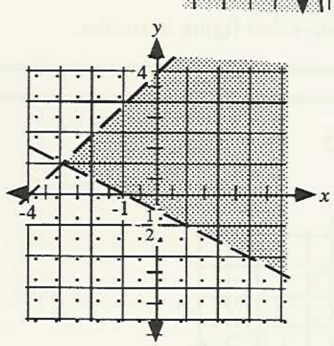
17. $\frac{5}{6}x - 3y \ge 9$
Graph $5x - 18y = 54$
Intercepts: $x = 10\frac{4}{5}$, $y = -3$
Test Point (0, 0): False



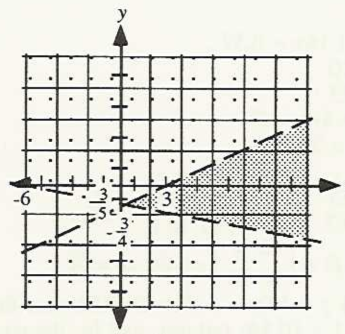
21. $1.5 > x + 0.1y$
$10x + y = 15$
Intercepts: $x = 1\frac{1}{2}$, $y = 15$
Test Point (0, 0): True



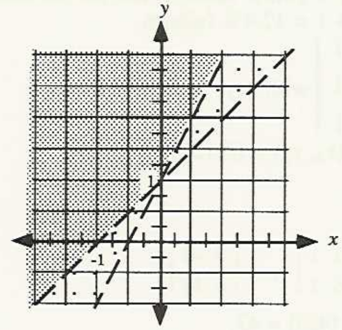
25.



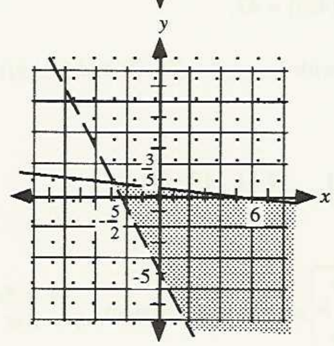
29.



33.



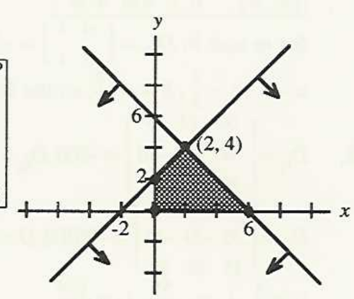
37.



41. $-x + y \le 2$
$x + y \le 6$
$P = 2x + y$

Solution:
$P = 12$ at (6,0)

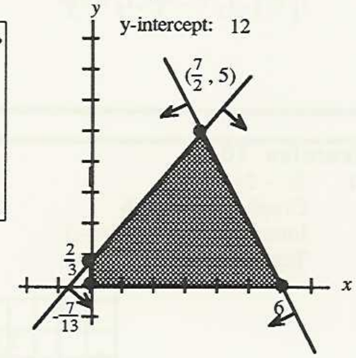
$2x + y = P$		
x	y	P
0	0	0
0	2	2
2	4	8
6	0	**12**



45. $-26x + 21y \le 14$
$2x + y \le 12$
$P = 2x + \frac{1}{3}y$

Solution:
$P = 12$ at (6,0)

$x + \frac{1}{3}y = P$		
x	y	P
0	0	0
0	$\frac{2}{3}$	$\frac{2}{9}$
$\frac{7}{2}$	5	$9\frac{2}{3}$
6	0	**12**

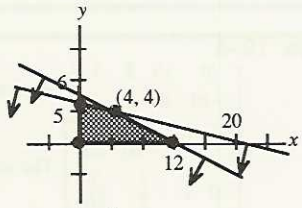


49. $x + 4y \le 20$
$2x + y \le 12$
$P = \frac{1}{3}x + \frac{1}{2}y$

Solution:
$P = 4$ at $(12,0)$

$\frac{1}{3}x + \frac{1}{2}y = P$

x	y	P
0	0	0
0	5	$2\frac{1}{2}$
4	4	$3\frac{1}{3}$
12	0	**4**

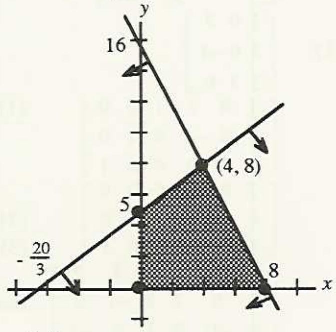


53. $-3x + 4y \le 20$
$2x + y \le 16$
$P = -\frac{1}{2}x + 2y$

Solution:
$P = 14$ at
$(4, 8)$

$-\frac{1}{2}x + 2y = P$

x	y	P
0	0	0
0	5	10
4	8	**14**
8	0	–4

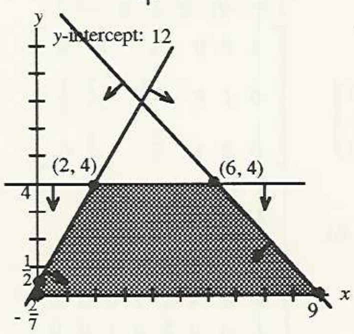


57. $-7x + 4y \le$
2
$y \le 4$
$\frac{4}{3}x + y \le 12$
$P = 2x + y$

Solution:
$P = 18$ at
$(9, 0)$

$2x + y = P$

0	0	0
0	$\frac{1}{2}$	$\frac{1}{2}$
2	4	8
6	4	16
9	0	**18**

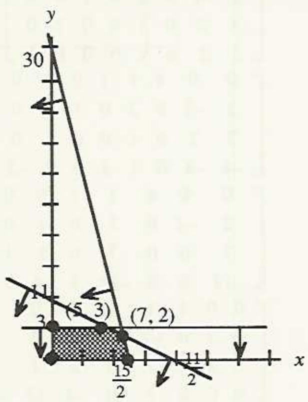


61. $0x + y \le 3$
$\frac{1}{2}x + y \le \frac{11}{2}$
$4x + y \le 30$
$P = \frac{1}{5}x - \frac{1}{8}y$

Solution:
$P = 1\frac{1}{2}$ at
$(7\frac{1}{2}, 0)$

$\frac{1}{5}x - \frac{1}{8}y = P$

x	y	P
0	0	0
0	3	$-\frac{8}{3}$
5	3	$\frac{5}{8}$
7	2	$1\frac{3}{20}$
$\frac{15}{2}$	0	$\mathbf{1\frac{1}{2}}$

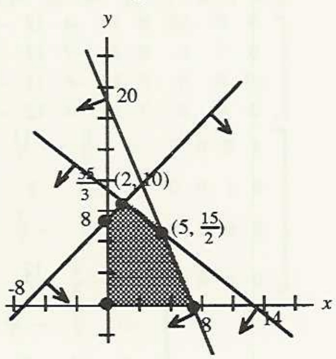


65. $-x + y \le 8$
$5x + 6y \le$
70
$5x + 2y \le$
40
$P = 3x + y$

Solution:
$P = 24$ at
$(8, 0)$

$3x + y = P$

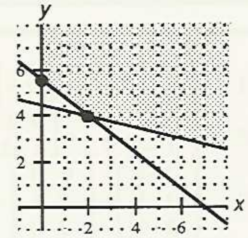
x	y	P
0	0	0
0	8	8
2	10	16
5	$\frac{15}{2}$	$22\frac{1}{2}$
8	0	**24**



69. $x + 4y \ge 18$
$4x + 5y \ge 28$
$C = 2x + 3y$

x	y	C
0	$\frac{28}{5}$	$16\frac{4}{5}$
2	4	16

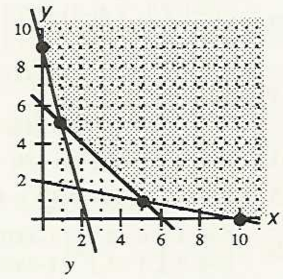
minimum is 16 at
$(2,4)$



73. $4x + y \ge 9$
$x + y \ge 6$
$x + 5y \ge 10$
$C = x + 2y$

x	y	C
0	9	18
1	5	11
5	1	7
10	0	10

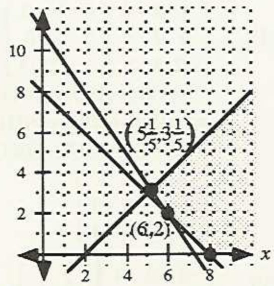
minimum is 7
at $(5,1)$



77. $-x + y \ge -2$
$3x + 2y \ge 22$
$x + y \ge 8$
$C = 3x - y$

x	y	C
$\frac{26}{5}$	$\frac{16}{5}$	$\frac{62}{5}$
6	2	16
8	0	24

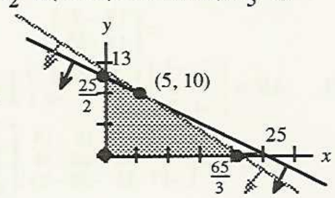
minimum is $12\frac{2}{5}$
at $\left(5\frac{1}{5}, 3\frac{1}{5}\right)$



81. Let x be the number of 12–worker/2–machine crews, and y be the number of 20–worker/4–machine crews. Then the number of workers on both types of crews is $12x + 20y$, and this must be less than or equal to 260. The number of machines on both types of crews is $2x + 4y$, and this must be less than or equal to 50. The amount of coal produced is $13x + 25y$. The system is

$$12x + 20y \le 260$$
$$2x + 4y \le 50,$$
$$C = 13x + 25y.$$

$C(0, 0) = 0$, $C(0, 12\frac{1}{2}) = 312\frac{1}{2}$, $C(5, 10) = 315$, $C(21\frac{2}{3}, 0) =$ $281\frac{2}{3}$. C is maximized at 315 tons with 5 type A crews and 10 type B crews.



Exercise 10–5

1. $\begin{bmatrix} -1 & 3 \\ -2 & 5 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 1 & -5 \end{bmatrix} = \begin{bmatrix} -1+3 & 3-2 \\ -2+1 & 5-5 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$

5. $\begin{bmatrix} -1 & -2 & -1 \\ 3 & -2 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 3 & -2 \\ 1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} -1+0 & -2+3 & -1-2 \\ 3+1 & -2-2 & 5-3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & -3 \\ 4 & -4 & 2 \end{bmatrix}$

9. $\begin{bmatrix} -1 & -2 \\ -1 & 3 \\ -2 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ -2 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} -1+0 & -2+3 \\ -1-2 & 3+1 \\ -2-2 & 5-3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -3 & 4 \\ -4 & 2 \end{bmatrix}$

13. $4\begin{bmatrix} -1 & 2 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 4(-1) & 4(2) \\ 4(4) & 4(5) \end{bmatrix} = \begin{bmatrix} -4 & 8 \\ 16 & 20 \end{bmatrix}$

17. $\frac{1}{2}\begin{bmatrix} 4 & 1 & -2 \\ 2 & 6 & -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(4) & \frac{1}{2}(1) & \frac{1}{2}(-2) \\ \frac{1}{2}(2) & \frac{1}{2}(6) & \frac{1}{2}(-2) \end{bmatrix} = \begin{bmatrix} 2 & \frac{1}{2} & -1 \\ 1 & 3 & -1 \end{bmatrix}$

21. $3(-2) + (-4)(5) = -26$

25. $\sqrt{2}\sqrt{8} + \frac{1}{3}(6) - 5(\frac{\pi}{5}) = 4 + 2 - \pi = 6 - \pi$

29. $\begin{bmatrix} -1 & 1 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ 1 & -5 \end{bmatrix} = \begin{bmatrix} (-1)(0)+(1)(1) & (-1)(-2)+(1)(-5) \\ (-2)(0)+(5)(1) & (-2)(-2)+(5)(-5) \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 5 & -21 \end{bmatrix}$

33. $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 3 \\ -2 & 5 & 2 \end{bmatrix}\begin{bmatrix} 0 & 3 & -1 \\ 5 & 2 & 0 \\ 1 & -2 & -3 \end{bmatrix}$

$= \begin{bmatrix} (1)(0)+(2)(5)+(-1)(1) & (1)(3)+(2)(2)+(-1)(-2) & (1)(-1)+(2)(0)+(-1)(-3) \\ (0)(0)+(2)(5)+(3)(1) & (0)(3)+(2)(2)+(3)(-2) & (0)(-1)+(2)(0)+(3)(-3) \\ (-2)(0)+(5)(5)+(2)(1) & (-2)(3)+(5)(2)+(2)(-2) & (-2)(-1)+(5)(0)+(2)(-3) \end{bmatrix}$

$= \begin{bmatrix} 9 & 9 & 2 \\ 13 & -2 & -9 \\ 27 & 0 & -4 \end{bmatrix}$

37. $\begin{bmatrix} -1 & -2 & -1 \\ 3 & -2 & 5 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ -2 & 1 \\ -2 & -3 \end{bmatrix}$

$= \begin{bmatrix} (-1)(0)+(-2)(-2)+(-1)(-2) & (-1)(3)+(-2)(1)+(-1)(-3) \\ (3)(0)+(-2)(-2)+(5)(-2) & (3)(3)+(-2)(1)+(5)(-3) \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ -6 & -8 \end{bmatrix}$

41. $\begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} -4 & -21 & 10 \\ 1 & 3 & -4 \end{bmatrix}$

$= \begin{bmatrix} (1)(-4)+(-3)(1) & (1)(-21)+(-3)(3) & (1)(10)+(-3)(-4) \\ (-1)(-4)+(4)(1) & (-1)(-21)+(4)(3) & (-1)(10)+(4)(-4) \end{bmatrix} = \begin{bmatrix} -7 & -30 & 22 \\ 8 & 33 & -26 \end{bmatrix}$

45. $\begin{bmatrix} 2 & 3 & -1 \\ 4 & \frac{1}{2} & 8 \end{bmatrix}\begin{bmatrix} -4 & -6 \\ 10 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} (2)(-4)+(3)(10)+(-1)(3) & (2)(-6)+(3)(2)+(-1)(-4) \\ (4)(-4)+(\frac{1}{2})(10)+(8)(3) & (4)(-6)+(\frac{1}{2})(2)+(8)(-4) \end{bmatrix}$

$= \begin{bmatrix} 19 & -2 \\ 13 & -55 \end{bmatrix}$

49. $AB = \begin{bmatrix} 2 & -1 \\ 4 & 6 \\ -4 & 5 \end{bmatrix}\begin{bmatrix} 3 & -5 & 2 & 7 \\ -1 & 3 & -6 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -13 & 10 & 13 \\ 6 & -2 & -28 & 34 \\ -17 & 35 & -38 & -23 \end{bmatrix}$,

$(AB)C = \begin{bmatrix} 7 & -13 & 10 & 13 \\ 6 & -2 & -28 & 34 \\ -17 & 35 & -38 & -23 \end{bmatrix}\begin{bmatrix} 3 & 0 & -1 \\ 4 & 2 & 0 \\ 2 & 5 & -1 \\ 8 & 3 & -4 \end{bmatrix} = \begin{bmatrix} 93 & 63 & -69 \\ 226 & -42 & -114 \\ -171 & -189 & 147 \end{bmatrix}$

$BC = \begin{bmatrix} 3 & -5 & 2 & 7 \\ -1 & 3 & -6 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 & -1 \\ 4 & 2 & 0 \\ 2 & 5 & -1 \\ 8 & 3 & -4 \end{bmatrix} = \begin{bmatrix} 49 & 21 & -33 \\ 5 & -21 & 3 \end{bmatrix}$,

$A(BC) = \begin{bmatrix} 2 & -1 \\ 4 & 6 \\ -4 & 5 \end{bmatrix}\begin{bmatrix} 49 & 21 & -33 \\ 5 & -21 & 3 \end{bmatrix} = \begin{bmatrix} 93 & 63 & -69 \\ 226 & -42 & -114 \\ -171 & -189 & 147 \end{bmatrix}$, so

$(AB)C = A(BC)$

53. $\begin{bmatrix} -3 & \frac{1}{2} \\ 2 & 3 \end{bmatrix}$

$\begin{bmatrix} -3 & \frac{1}{2} & 1 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix}$ Multiply [1] by 2.

$\begin{bmatrix} -6 & 1 & 2 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix}$ [1] <= 3[2] + 1[1]

$\begin{bmatrix} 0 & 10 & 2 & 3 \\ 2 & 3 & 0 & 1 \end{bmatrix}$ [2] <= 3[1] + −10[2]

$\begin{bmatrix} 0 & 10 & 2 & 3 \\ -20 & 0 & 6 & -1 \end{bmatrix}$ Rearrange rows and set coefficients to 1.

$\begin{bmatrix} 1 & 0 & -\frac{3}{10} & \frac{1}{20} \\ 0 & 1 & \frac{1}{5} & \frac{3}{10} \end{bmatrix}$ The inverse of $\begin{bmatrix} -3 & \frac{1}{2} \\ 2 & 3 \end{bmatrix}$ is $\begin{bmatrix} -\frac{3}{10} & \frac{1}{20} \\ \frac{1}{5} & \frac{3}{10} \end{bmatrix}$.

57. $\begin{bmatrix} 1 & 0 & 3 \\ 2 & 0 & -1 \\ 2 & 3 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 3 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 0 & 1 \end{bmatrix}$ [1] <= 3[2] + 1[1]

$\begin{bmatrix} 7 & 0 & 0 & 1 & 3 & 0 \\ 2 & 0 & -1 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 0 & 1 \end{bmatrix}$ [2] <= 2[1] + −7[2]
[3] <= 2[1] + −7[3]

$\begin{bmatrix} 7 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 7 & 2 & -1 & 0 \\ 0 & -21 & 0 & 2 & 6 & -7 \end{bmatrix}$ Rearrange rows and set coefficients to 1.

$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{7} & \frac{3}{7} & 0 \\ 0 & 1 & 0 & -\frac{2}{21} & -\frac{2}{7} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{2}{7} & -\frac{1}{7} & 0 \end{bmatrix}$ The inverse is $\begin{bmatrix} \frac{1}{7} & \frac{3}{7} & 0 \\ -\frac{2}{21} & -\frac{2}{7} & \frac{1}{3} \\ \frac{2}{7} & -\frac{1}{7} & 0 \end{bmatrix}$.

61. $\begin{bmatrix} 0 & 0 & 4 & 1 \\ 2 & -1 & 0 & 3 \\ 3 & 2 & 0 & 1 \\ 2 & 2 & 6 & 1 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 & 4 & 1 & 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 3 & 0 & 1 & 0 & 0 \\ 3 & 2 & 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 2 & 6 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$ [4] <= 3[1] + −2[4]

$\begin{bmatrix} 0 & 0 & 4 & 1 & 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 3 & 0 & 1 & 0 & 0 \\ 3 & 2 & 0 & 1 & 0 & 0 & 1 & 0 \\ -4 & -4 & 0 & 1 & 3 & 0 & 0 & -2 \end{bmatrix}$ [3] <= 2[2] + 1[3]
[4] <= −4[2] + 1[4]

$\begin{bmatrix} 0 & 0 & 4 & 1 & 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 3 & 0 & 1 & 0 & 0 \\ 7 & 0 & 0 & 7 & 0 & 2 & 1 & 0 \\ -12 & 0 & 0 & -11 & 3 & -4 & 0 & -2 \end{bmatrix}$ [2] <= 2[3] + −7[2]
[4] <= 12[3] + 7[4]
[1] <= 1[4] + −7[1]
[2] <= 1[4] + 1[2]
[3] <= 1[4] + −1[3]

$\begin{bmatrix} 0 & 0 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & 0 & -7 & 0 & -3 & 2 & 0 \\ 7 & 0 & 0 & 7 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 7 & 21 & -4 & 12 & -14 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 & -28 & 0 & 14 & -4 & 12 & -14 \\ 0 & 7 & 0 & 0 & 21 & -7 & 14 & -14 \\ -7 & 0 & 0 & 0 & 21 & -6 & 11 & -14 \\ 0 & 0 & 0 & 7 & 21 & -4 & 12 & -14 \end{bmatrix}$ Rearrange rows and set coefficients to 1.

$\begin{bmatrix} 1 & 0 & 0 & 0 & -3 & \frac{6}{7} & -\frac{11}{7} & 2 \\ 0 & 1 & 0 & 0 & 3 & -1 & 2 & -2 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{7} & -\frac{3}{7} & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 3 & -\frac{4}{7} & \frac{12}{7} & -2 \end{bmatrix}$

The inverse is $\begin{bmatrix} -3 & \frac{6}{7} & -\frac{11}{7} & 2 \\ 3 & -1 & 2 & -2 \\ -\frac{1}{2} & \frac{1}{7} & -\frac{3}{7} & \frac{1}{2} \\ 3 & -\frac{4}{7} & \frac{12}{7} & -2 \end{bmatrix}$.

65. $-3x + \frac{1}{2}y = -7$
$2x + 3y = -2$

$$\begin{bmatrix} -\frac{3}{10} & \frac{1}{20} \\ \frac{1}{5} & \frac{3}{10} \end{bmatrix}\begin{bmatrix} -7 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}; x = 2, y = -2.$$

69. $x + 3z = -6$
$2x - z = -5$
$2x + 3y = -4$

$$\begin{bmatrix} \frac{1}{7} & \frac{3}{7} & 0 \\ -\frac{2}{21} & -\frac{2}{7} & \frac{1}{3} \\ \frac{2}{7} & -\frac{1}{7} & 0 \end{bmatrix}\begin{bmatrix} -6 \\ -5 \\ -4 \end{bmatrix} = \begin{bmatrix} -3 \\ \frac{2}{3} \\ -1 \end{bmatrix}; x = -3, y = \frac{2}{3}, z = -1.$$

73. $4z + w = -2$
$2x - y + 3w = 5$
$3x + 2y + w = 11$
$2x + 2y + 6z + w = 4$

$$\begin{bmatrix} -3 & \frac{6}{7} & -\frac{11}{7} & 2 \\ 3 & -1 & 2 & -2 \\ -\frac{1}{2} & \frac{1}{7} & -\frac{3}{7} & \frac{1}{2} \\ 3 & -\frac{4}{7} & \frac{12}{7} & -2 \end{bmatrix}\begin{bmatrix} -2 \\ 5 \\ 11 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \\ 2 \end{bmatrix};$$

$x = 1, y = 3, z = -1, w = 2.$

77. $-2x + y = 1$
$4x - y = 0$
The inverse of $\begin{bmatrix} -2 & 1 \\ 4 & -1 \end{bmatrix}$ is $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 2 & 1 \end{bmatrix}$.

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 2 \end{bmatrix}; x = \frac{1}{2}, y = 2.$$

81. $-1x - y - 2z = 1$
$x + y - 4z = -4$
$-\frac{1}{2}y + 2z = 3$

The inverse of $\begin{bmatrix} -1 & -1 & -2 \\ 1 & 1 & -4 \\ 0 & -\frac{1}{2} & 2 \end{bmatrix}$ is $\begin{bmatrix} 0 & 1 & 2 \\ -\frac{2}{3} & -\frac{2}{3} & -2 \\ -\frac{1}{6} & -\frac{1}{6} & 0 \end{bmatrix}$.

$$\begin{bmatrix} 0 & 1 & 2 \\ -\frac{2}{3} & -\frac{2}{3} & -2 \\ -\frac{1}{6} & -\frac{1}{6} & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ \frac{1}{2} \end{bmatrix}; x = 2, y = -4, z = \frac{1}{2}.$$

85. $2\begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} - (-3)\begin{bmatrix} 0 & 3 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ -2 & 8 \end{bmatrix} + \begin{bmatrix} 0 & 9 \\ 12 & -3 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 10 & 5 \end{bmatrix}$

89. $$\begin{bmatrix} 1&1&0&1&1 \\ 0&0&1&0&1 \\ 0&1&1&0&1 \\ 1&1&1&1&1 \end{bmatrix}\begin{bmatrix} 1&0&0&1 \\ 1&0&1&1 \\ 0&1&1&1 \\ 1&0&0&1 \\ 1&1&1&1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+0+1+1 & 0+0+0+0+1 & 0+1+0+0+1 & 1+1+0+1+1 \\ 0+0+0+0+1 & 0+0+1+0+1 & 0+0+1+0+1 & 0+0+1+0+1 \\ 0+1+0+0+1 & 0+0+1+0+1 & 0+1+1+0+1 & 0+1+1+0+1 \\ 1+1+0+1+1 & 0+0+1+0+1 & 0+1+1+0+1 & 1+1+1+1+1 \end{bmatrix} = \begin{bmatrix} 4&1&2&4 \\ 1&2&2&2 \\ 2&2&3&3 \\ 4&2&3&5 \end{bmatrix}$$

93. $A^3 = AA^2 = \begin{bmatrix} 0&1&1&1 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&1&1&0 \end{bmatrix}\begin{bmatrix} 1&1&1&0 \\ 0&1&1&1 \\ 0&0&0&0 \\ 1&0&0&0 \end{bmatrix} = \begin{bmatrix} 1&\boxed{1}&1&1 \\ 1&1&1&0 \\ 0&0&0&0 \\ 0&1&1&1 \end{bmatrix}$; 1 path of length 3 from node 1 to node 2.

97. $L = \begin{bmatrix} 0 & 4 & 3 \\ .229 & 0 & 0 \\ 0 & .125 & 0 \end{bmatrix}; LV = \begin{bmatrix} 7000 \\ 229 \\ 125 \end{bmatrix}$

$$L^2V = \begin{bmatrix} 0 & 4 & 3 \\ .229 & 0 & 0 \\ 0 & .125 & 0 \end{bmatrix}\begin{bmatrix} 7000 \\ 229 \\ 125 \end{bmatrix} = \begin{bmatrix} 1291 \\ 1603 \\ 29 \end{bmatrix}$$

$$L^3V = \begin{bmatrix} 0 & 4 & 3 \\ .229 & 0 & 0 \\ 0 & .125 & 0 \end{bmatrix}\begin{bmatrix} 1291 \\ 1603 \\ 29 \end{bmatrix} = \begin{bmatrix} 6498 \\ 296 \\ 200 \end{bmatrix}$$

Thus there are 6498, 296, and 200 females in each stage after three life cycles.

Chapter 10 Review

1. [1] $3x + 2y = -6$
[2] $-\frac{3}{2}x + y = 6$
Clear denominators; multiply [2] by 2:
[1] $3x + 2y = -6$
[2] $-3x + 2y = 12$
Eliminate x:
[3] $4y = 6$ $\Leftarrow$ [1] + [2]
$\mathbf{y = \frac{3}{2}}$
Insert value of y into [1]:
[1] $3x + 2(\frac{3}{2}) = -6$
$\mathbf{x = -3}$

3. [1] $-x + 4y = 20$
[2] $2x + y = -22$
Multiply [1] by 2:
[1] $-2x + 8y = 40$
[2] $2x + y = -22$
Eliminate x:
[3] $9y = 18$ $\Leftarrow$ 4[1] + 3[2]
$\mathbf{y = 2}$
Insert value of y into [2]:
[2] $2x + 2 = -22$
$\mathbf{x = -12}$

5. [1] $x - 5z = -7$
[2] $-2x + y + 2z = 9$
[3] $5x + 2y = -4$
Use [2] to eliminate y from [3]; it is already eliminated from [1]. Put result in [4]:
[4] $9x - 4z = -22$ $\Leftarrow$ −2[2] + [3]
[1] $x - 5z = -7$
Equations [1] and [4] involve only x and z; eliminate x:
[5] $41z = 41$ $\Leftarrow$ −9[1] + [4]
$\mathbf{z = 1}$
Insert value of z into [1]:
[1] $x - 5(1) = -7$
$\mathbf{x = -2}$
Insert value of x and z into [2]:
[2] $-2(-2) + y + 2(1) = 9$
$\mathbf{y = 3}$

7. [1] $x + y - z = 7$
[2] $-2x + 4y + 2w = 4$
[3] $-2y - 3z - 2w = 0$
[4] $2x - 5z + 5w = 21$
Eliminate x from equations [2] and [4]:
[5] $6y - 2z + 2w = 18$ $\Leftarrow$ 2[1] + [2]
[6] $4y - 5z + 7w = 25$ $\Leftarrow$ [2] + [4]
[3] $-2y - 3z - 2w = 0$
Equations [3], [5], [6] involve only y, z, and w. Eliminate y using [3]:
[7] $-11z - 4w = 18$ $\Leftarrow$ 3[3] + [5]
[8] $-11z + 3w = 25$ $\Leftarrow$ 2[3] + [6]
Now [7] and [8] only involve 2 variables, z and w:
[9] $7w = 7$ $\Leftarrow$ −[7] + [8]
$\mathbf{w = 1}$
Insert value of w into [8]:
[8] $-11z + 3(1) = 25$
$\mathbf{z = -2}$
Insert value of w and z into [4]:
[4] $2x - 5(-2) + 5(1) = 21$
$\mathbf{x = 3}$
Insert value of z and w into [3]:
[3] $-2y - 3(-2) - 2(1) = 0$
$\mathbf{y = 2}$

9. L = length, W = width; $\frac{L}{W} = \frac{8}{5}$, so $5L - 8W = 0$. Also $2L + 2W = 182$. Solving the system $\begin{matrix} 2L + 2W = 182 \\ 5L - 8W = 0 \end{matrix}$ we find $L = 56$ cm, $W = 35$ cm.

11. x = amount at 6%, and y amount at 12%. Then $\begin{matrix} 0.06x + 0.12y = 1530 \\ x + y = 15000 \end{matrix}$, which we solve to find $x = \$4{,}500$ and $y = \$10{,}500$.

13. $-2x + 5y = 7$
$2x + 3y = 9$

$\begin{bmatrix} -2 & 5 & 7 \\ 2 & 3 & 9 \end{bmatrix}$ [2] ⇐ [1] + [2]

$\begin{bmatrix} -2 & 5 & 7 \\ 0 & 8 & 16 \end{bmatrix}$ Divide [2] by 8.

$\begin{bmatrix} -2 & 5 & 7 \\ 0 & 1 & 2 \end{bmatrix}$ [1] ⇐ 5[2] − [1]

$\begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$ Rearrange rows and set coefficients to 1.

$\begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & 2 \end{bmatrix}$ Solution: $(\frac{3}{2}, 2)$

15. $-5x + y + 2z = 4$
$4x - y - z = -5$
$\frac{1}{2}x + 2y - 5z = 14$

$\begin{bmatrix} -5 & 1 & 2 & 4 \\ 4 & -1 & -1 & -5 \\ \frac{1}{2} & 2 & -5 & 14 \end{bmatrix}$ Multiply [3] by 2.

$\begin{bmatrix} -5 & 1 & 2 & 4 \\ 4 & -1 & -1 & -5 \\ 1 & 4 & -10 & 28 \end{bmatrix}$ [2] ⇐ [1] + [2]
[3] ⇐ 4[1] − [3]

$\begin{bmatrix} -5 & 1 & 2 & 4 \\ -1 & 0 & 1 & -1 \\ -21 & 0 & 18 & -12 \end{bmatrix}$ Divide [3] by 3.

$\begin{bmatrix} -5 & 1 & 2 & 4 \\ -1 & 0 & 1 & -1 \\ -7 & 0 & 6 & -4 \end{bmatrix}$ [1] ⇐ 2[2] − [1]
[3] ⇐ 6[2] − [3]

$\begin{bmatrix} 3 & -1 & 0 & -6 \\ -1 & 0 & 1 & -1 \\ 1 & 0 & 0 & -2 \end{bmatrix}$ [1] ⇐ 3[3] − [1]
[2] ⇐ [3] + [2]

$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 1 & 0 & 0 & -2 \end{bmatrix}$ Rearrange rows and set coefficients to 1.

$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \end{bmatrix}$ Solution: $(-2, 0, -3)$

17. $x - 3z + 5w = -9$
$-x + z + 3w = -7$
$3x - 3y + z = 15$
$-5x + y - 5z + 5w = -19$

$\begin{bmatrix} 1 & 0 & -3 & 5 & -9 \\ -1 & 0 & 1 & 3 & -7 \\ 3 & -3 & 1 & 0 & 15 \\ -5 & 1 & -5 & 5 & -19 \end{bmatrix}$ [3] ⇐ 3[4] + [3]

$\begin{bmatrix} 1 & 0 & -3 & 5 & -9 \\ -1 & 0 & 1 & 3 & -7 \\ -12 & 0 & -14 & 15 & -42 \\ -5 & 1 & -5 & 5 & -19 \end{bmatrix}$ [2] ⇐ [1] + [2]
[3] ⇐ 12[1] + [3]
[4] ⇐ 5[1] + [4]

$\begin{bmatrix} 1 & 0 & -3 & 5 & -9 \\ 0 & 0 & -2 & 8 & -16 \\ 0 & 0 & -50 & 75 & -150 \\ 0 & 1 & -20 & 30 & -64 \end{bmatrix}$ Divide [2] by 2 and [3] by 25.

$\begin{bmatrix} 1 & 0 & -3 & 5 & -9 \\ 0 & 0 & -1 & 4 & -8 \\ 0 & 0 & -2 & 3 & -6 \\ 0 & 1 & -20 & 30 & -64 \end{bmatrix}$ [1] ⇐ −3[2] + [1]
[3] ⇐ −2[2] + [3]
[4] ⇐ −20[2] + [4]

$\begin{bmatrix} 1 & 0 & 0 & -7 & 15 \\ 0 & 0 & -1 & 4 & -8 \\ 0 & 0 & 0 & -5 & 10 \\ 0 & 1 & 0 & -50 & 96 \end{bmatrix}$ Divide [3] by 5.

$\begin{bmatrix} 1 & 0 & 0 & -7 & 15 \\ 0 & 0 & -1 & 4 & -8 \\ 0 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & -50 & 96 \end{bmatrix}$ [1] ⇐ −7[3] + [1]
[2] ⇐ 4[3] + [2]
[4] ⇐ −50[3] + [4]

$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & 0 & -4 \end{bmatrix}$ Rearrange rows and set coefficients to 1.

$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$ Solution: $(1, -4, 0, -2)$

19. $25i_1 + 20i_2 + 5i_3 = 50$
$40i_1 - 20i_2 - 10i_3 = 40$
$5i_1 + 10i_2 + 5i_3 = 45$

$$\begin{bmatrix} 25 & 20 & 5 & 50 \\ 40 & -20 & -10 & 40 \\ 5 & 10 & 5 & 45 \end{bmatrix} \to \begin{bmatrix} 5 & 4 & 1 & 10 \\ 8 & -4 & -2 & 8 \\ 1 & 2 & 1 & 9 \end{bmatrix} \to \begin{bmatrix} 0 & -6 & -4 & -35 \\ 0 & -10 & -5 & -32 \\ 1 & 2 & 1 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -10 & 0 & 47 \\ 0 & -10 & -5 & -32 \\ 5 & 0 & 0 & 13 \end{bmatrix} \to \begin{bmatrix} 0 & 10 & 0 & -47 \\ 0 & 0 & 5 & 79 \\ 5 & 0 & 0 & 13 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & \frac{13}{5} \\ 0 & 1 & 0 & -\frac{47}{10} \\ 0 & 0 & 1 & \frac{79}{5} \end{bmatrix}$$

$i_1 = 2\frac{3}{5}$, $i_2 = -4\frac{7}{10}$, $i_3 = 15\frac{4}{5}$

21. $-3x - 4y = 0$
$x + 9y = 4$
$D = \begin{vmatrix} -3 & -4 \\ 1 & 9 \end{vmatrix} = -23$; $D_x = \begin{vmatrix} 0 & -4 \\ 4 & 9 \end{vmatrix} = 16$

$Dy = \begin{vmatrix} -3 & 0 \\ 1 & 4 \end{vmatrix} = -12$; $x = -\frac{16}{23}$, $y = \frac{12}{23}$

23. $x + 8y + 3z = -4$
$x - 3y = 5$
$-x + 9y + 7z = -6$
; $D = \begin{vmatrix} 1 & 8 & 3 \\ 1 & -3 & 0 \\ -1 & 9 & 7 \end{vmatrix} = -59$;

$D_x = \begin{vmatrix} -4 & 8 & 3 \\ 5 & -3 & 0 \\ -6 & 9 & 7 \end{vmatrix} = -115$; $Dy = \begin{vmatrix} 1 & -4 & 3 \\ 1 & 5 & 0 \\ -1 & -6 & 7 \end{vmatrix} = 60$;

$D_z = \begin{vmatrix} 1 & 8 & -4 \\ 1 & -3 & 5 \\ -1 & 9 & -6 \end{vmatrix} = -43$

$x = \frac{115}{59}$, $y = -\frac{60}{59}$, $z = \frac{43}{59}$

25. $2y + 3z - w = 2$
$-x + 2y + 5z = 0$
$-4x - 2z - 4w = -2$
$-4y - 4w = 1$
; $D = \begin{vmatrix} 0 & 2 & 3 & -1 \\ -1 & 2 & 5 & 0 \\ -4 & 0 & -2 & -4 \\ 0 & -4 & 0 & -4 \end{vmatrix} = 216$

$D_x = \begin{vmatrix} 2 & 2 & 3 & -1 \\ 0 & 2 & 5 & 0 \\ -2 & 0 & -2 & -4 \\ 1 & -4 & 0 & -4 \end{vmatrix} = 276$; $Dy = \begin{vmatrix} 0 & 2 & 3 & -1 \\ -1 & 0 & 5 & 0 \\ -4 & -2 & -2 & -4 \\ 0 & 1 & 0 & -4 \end{vmatrix} = 118$

$D_z = \begin{vmatrix} 0 & 2 & 2 & -1 \\ -1 & 2 & 0 & 0 \\ -4 & 0 & -2 & -4 \\ 0 & -4 & 1 & -4 \end{vmatrix} = 8$; $D_w = \begin{vmatrix} 0 & 2 & 3 & 2 \\ -1 & 2 & 5 & 0 \\ -4 & 0 & -2 & -2 \\ 0 & -4 & 0 & 1 \end{vmatrix} = -172$

$x = \frac{23}{18}$, $y = \frac{59}{108}$, $z = \frac{1}{27}$, $w = -\frac{43}{54}$

27. $M = \begin{vmatrix} 2 & -1 & 3 & -1 & 0 \\ 1 & 1 & 0 & -2 & 1 \\ 0 & -1 & -1 & 0 & 0 \\ 3 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 2 & -1 & -1 & 0 \\ 1 & 1 & -2 & 1 \\ 0 & -1 & 0 & 0 \\ 3 & 0 & 1 & 1 \end{vmatrix} + (-1)\begin{vmatrix} 2 & -1 & 3 & -1 \\ 1 & 1 & 0 & -2 \\ 0 & -1 & -1 & 0 \\ 3 & 0 & -1 & 1 \end{vmatrix}$

$= -8 - (-36) = 28$

$M_D = \begin{vmatrix} 2 & -1 & 3 & 5 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ \mathbf{0} & \mathbf{-1} & \mathbf{-1} & \mathbf{10} & \mathbf{0} \\ 3 & 0 & -1 & -4 & 1 \\ 0 & 0 & 1 & -20 & -1 \end{vmatrix} =$

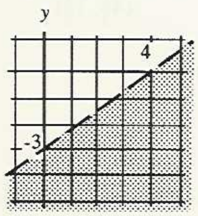
$-(-1)\begin{vmatrix} 2 & 3 & 5 & 0 \\ 1 & 0 & 0 & 1 \\ 3 & -1 & -4 & 1 \\ 0 & 1 & -20 & -1 \end{vmatrix} + (-1)\begin{vmatrix} 2 & -1 & 5 & 0 \\ 1 & 1 & 0 & 1 \\ 3 & 0 & -4 & 1 \\ 0 & 0 & -20 & -1 \end{vmatrix} - (10)\begin{pmatrix} 2 & -1 & 3 & 0 \\ 1 & 1 & 0 & 1 \\ 3 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$

$= 171 - 27 - 10(12) = 24$

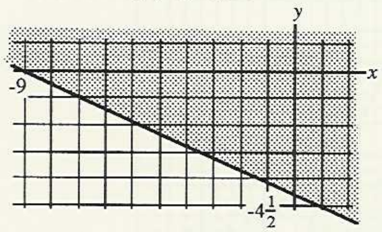
$D = \frac{M_D}{M} = \frac{24}{28} = \frac{6}{7}$

Complete solution: $(-\frac{58}{7}, -\frac{367}{28}, \frac{87}{28}, \frac{6}{7}, \frac{647}{28})$

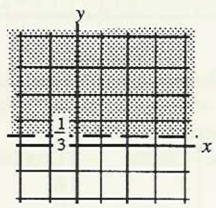
29. $3x - 4y > 12$
$3x - 4y = 12$
Intercepts:
$x = 4$, $y = -3$
Test Point
(0, 0): False



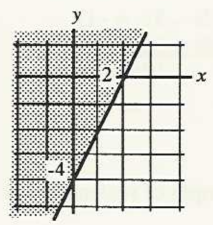
31. $x + 2y \geq -9$
$x + 2y = -9$
Intercepts: $x = -9$, $y = -4\frac{1}{2}$
Test Point (0, 0): True



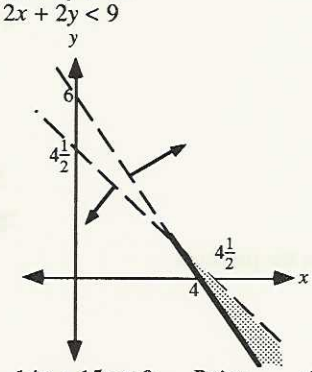
33. $6y > 2$
$6y = 2$
Intercepts:
$y = \frac{1}{3}$
Test Point
(0, 0): False



35. $2.4x - 1.2y \leq 4.8$
$2.4x - 1.2y = 4.8$
Intercepts:
$x = 2$, $y = -4$
Test Point
(0, 0): True



37. $3x + 2y \geq 12$
$2x + 2y < 9$

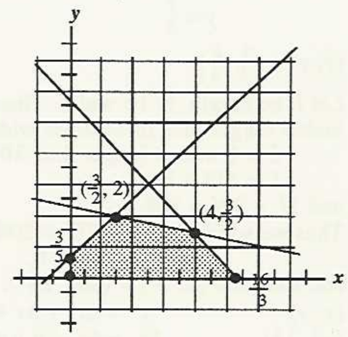


39. $-14x + 15y \leq 9$
$2x + 10y \leq 23$
$9x + 8y \leq 48$

$C = 4x + 2y$

Point	C
(0, 0)	0
$(0, \frac{3}{5})$	$1\frac{1}{5}$
$(\frac{3}{2}, 2)$	10
$(4, \frac{3}{2})$	19
$(\frac{16}{3}, 0)$	$\mathbf{21\frac{1}{3}}$

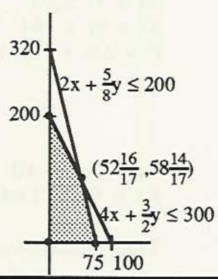
C is maximized for $x = 5\frac{1}{3}$, $y = 0$. Its value is $21\frac{1}{3}$.



41. x = #tables to produce, y = #chairs to produce; the objective function to be maximized is profit $C = 3x + y$. Four hours to assemble a table and one and one half to assemble a chair; 300 hours available: $4x + \frac{3}{2}y \leq 300$. Two hours to finish a table, five–eighths of an hour for a chair; 200 hours available: $2x + \frac{5}{8}y \leq 200$. The system is

$4x + \frac{3}{2}y \leq 300$
$2x + \frac{5}{8}y \leq 200$
$C = 3x + y$

Income is maximized at $225 by producing 75 tables and no chairs per run.



Point	$C = 3x + y$
(0, 0)	0
(75, 0)	**225**
$(52\frac{16}{17}, 58\frac{14}{17})$	$217\frac{11}{17}$
(0, 200)	200

43. $(3, -\frac{1}{4}) \cdot (-2, 5) = 3(-2) + (-\frac{1}{4})(5) = 4\frac{3}{4}$

45. $[\frac{1}{4}, 1, -2] \begin{bmatrix} 4 \\ -4 \\ 5 \end{bmatrix} = (\frac{1}{4})(4) + 1(-4) + (-2)(5) = -13$

47. There are an unlimited number of solutions; one is $(0, 0, \frac{1}{2}, 0)$.

49. $\begin{bmatrix} -x & 2 \\ 4y & -3 \end{bmatrix} \begin{bmatrix} 4x & 3 \\ y & 9 \end{bmatrix} = \begin{bmatrix} -4x^2+2y & -3x+18 \\ 16xy-3y & 12y-27 \end{bmatrix}$

51. $\begin{bmatrix} 2 & -5 \\ -3 & -1 \end{bmatrix}$

$\begin{bmatrix} 2 & -5 & 1 & 0 \\ -3 & -1 & 0 & 1 \end{bmatrix}$ $[2] \Leftarrow 3[1] + 2[2]$

$\begin{bmatrix} 2 & -5 & 1 & 0 \\ 0 & -17 & 3 & 2 \end{bmatrix}$ $[1] \Leftarrow -5[2] + 17[1]$

$\begin{bmatrix} 34 & 0 & 2 & -10 \\ 0 & -17 & 3 & 2 \end{bmatrix}$ Rearrange rows and set coefficients to 1.

$\begin{bmatrix} 1 & 0 & 1/17 & -5/17 \\ 0 & 1 & -3/17 & -2/17 \end{bmatrix}$. Thus the inverse is $\begin{bmatrix} \frac{1}{17} & -\frac{5}{17} \\ -\frac{3}{17} & -\frac{2}{17} \end{bmatrix}$.

53. $\begin{matrix} 2x - 3y = 1 \\ x + y = -2 \end{matrix}$ is $AX = B$, where $A = \begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$X = A^{-1}B = \begin{bmatrix} \frac{1}{5} & \frac{3}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, so $x = -1$, $y = -1$.

Chapter 10 Test

1. $$\begin{array}{rl} -2\,[2x - 3y &= -1] \\ \underline{4x + 9y} &\underline{= \ 8} \\ 15y &= 10 \\ y &= \frac{2}{3} \end{array} \qquad \begin{array}{rl} 3\,[2x - 3y &= -1] \\ \underline{4x + 9y} &\underline{= \ 8} \\ 10x &= 5 \\ x &= \frac{1}{2} \end{array}$$
$(x, y) = \left(\frac{1}{2}, \frac{2}{3}\right)$

3. Let L be length, W be width. Then "length of rectangle is 8 inches longer than three times width"
L = 8 inches longer than $3W$
$L = 3W + 8$
and $2L + 2W = 208$.
Thus we solve $\quad 2L + 2W = 208$
$\qquad L = 3W + 8,$
and find that $(L, W) = (80", 24")$.

5.

(x, y)	$y = ax^2 + bx + c$
$(-2, 13)$	$13 = 4a - 2b + c$
$(1, 4)$	$4 = a + b + c$
$(2, 9)$	$9 = 4a + 2b + c$

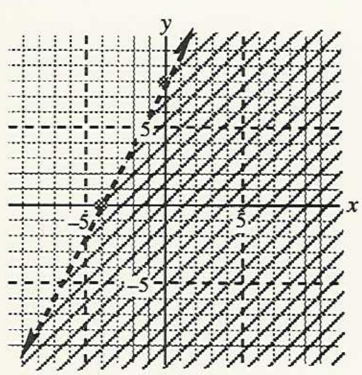
Solving this system gives (a, b, c) = (2, –1, 3), so the parabola is $y = 2x^2 - x + 3$.

7. $$\begin{bmatrix} 2 & -1 & 2 & 0 & 4 \\ 1 & 2 & 0 & -1 & -3 \\ 3 & -2 & 1 & 1 & 4 \\ 0 & 1 & 0 & -5 & 3 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 & 2 & -5 & 7 \\ 1 & 0 & 0 & 9 & -9 \\ 3 & 0 & 1 & -9 & 10 \\ 0 & 1 & 0 & -5 & 3 \end{bmatrix}$$
$$\begin{bmatrix} -4 & 0 & 0 & 13 & -13 \\ 1 & 0 & 0 & 9 & -9 \\ 3 & 0 & 1 & -9 & 10 \\ 0 & 1 & 0 & -5 & 3 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 49 & -49 \\ 1 & 0 & 0 & 9 & -9 \\ 0 & 0 & 1 & -36 & 37 \\ 0 & 1 & 0 & -5 & 3 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 9 & -9 \\ 0 & 0 & 1 & -36 & 37 \\ 0 & 1 & 0 & -5 & 3 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -2 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \quad (x, y, z, w) = (0, -2, 1, -1)$$

9. $$D = \begin{bmatrix} 1 & \frac{1}{3} \\ -1 & 2 \end{bmatrix} = \frac{7}{3};\ D_x = \begin{bmatrix} 4 & \frac{1}{3} \\ \frac{5}{4} & 2 \end{bmatrix} = \frac{91}{12}$$
$$D_y = \begin{bmatrix} 1 & 4 \\ -1 & \frac{5}{4} \end{bmatrix} = \frac{21}{4};\ x = \frac{D_x}{D} = \frac{13}{4};\ y = \frac{D_y}{D} = \frac{9}{4}.$$

11. $$\frac{1}{2}\begin{vmatrix} 3 & -1 & 1 \\ 6 & -3 & 1 \\ -2 & 8 & 1 \end{vmatrix} = \frac{1}{2}(17) = 8\tfrac{1}{2}.$$

13. $5y - 40 < 10x$

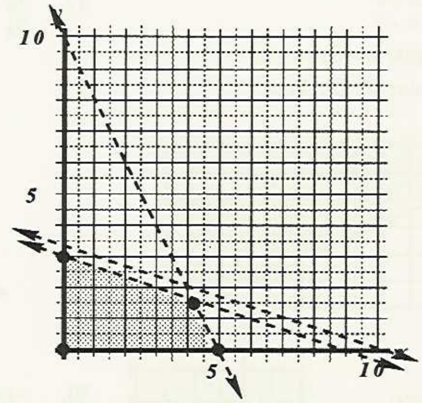


15. $2x + 6y \le 18$
$x + 3y \le 10$
$2x + y \le 10$
$C = x + 2y$

$$\left.\begin{array}{l} 2x + 6y \le 18 \\ 2x + y \le 10 \end{array}\right\} \qquad \left(4\tfrac{1}{5}, 1\tfrac{3}{5}\right)$$

Point	$C = x + 2y$
(0, 0)	0
(0, 3)	6
$\left(4\frac{1}{5}, 1\frac{3}{5}\right)$	$7\frac{2}{5}$
(5, 0)	5

Thus C is maximized at $x = 4\frac{1}{5}$, $y = 1\frac{3}{5}$, with a value of $7\frac{2}{5}$.



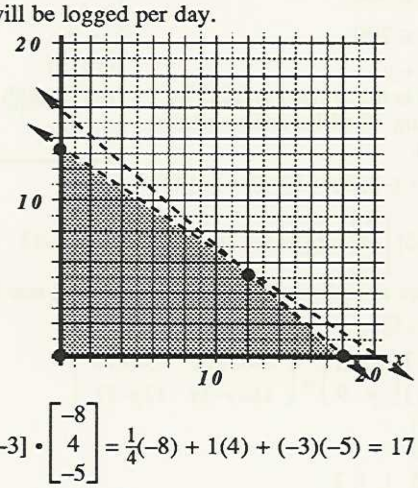
17. Let x be the number of crews of the first type, and y the number of crews of the second type, as shown in the table. Let P represent Productivity.

Crew Type	Chiefs	Loggers	Trees Logged
x	2	8	22
y	3	9	31
Available	40	144	

$2x + 3y \le 40$
$8x + 9y \le 144$
$P = 22x + 31y$ (to be maximized).

	Point	$P = 22x + 31y$
	(0, 0)	0
	$\left(0, 13\frac{1}{3}\right)$	$413\frac{1}{3}$
$\left.\begin{array}{l} 2x + 3y = 40 \\ 8x + 9y = 144 \end{array}\right\}$	$\left(12, 5\frac{1}{3}\right)$	$429\frac{1}{3}$
	(18, 0)	396

Thus productivity is maximized by using 12 of the first type of crews and $5\frac{1}{3}$ of the second type of crews. In this case, $429\frac{1}{3}$ trees will be logged per day.



19. $$\left[\tfrac{1}{4}, 1, -3\right] \cdot \begin{bmatrix} -8 \\ 4 \\ -5 \end{bmatrix} = \tfrac{1}{4}(-8) + 1(4) + (-3)(-5) = 17$$

21. $$\begin{bmatrix} 2 & -1 & 1 \\ 0 & 2 & 6 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 1 & 1 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 14 & -22 \end{bmatrix}$$

23. $\begin{bmatrix} 2 & -2 & 3 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & -3 & 2 & 0 & 0 & 1 \end{bmatrix}$ 1st row − 2(2nd row), 3rd row − 2nd row

$\begin{bmatrix} 0 & -2 & 5 & 1 & -2 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & -3 & 3 & 0 & -1 & 1 \end{bmatrix}$ 2(3rd row) − 3(1st row)

$\begin{bmatrix} 0 & -2 & 5 & 1 & -2 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -9 & -3 & 4 & 2 \end{bmatrix}$ Divide 3rd row by −9.

$\begin{bmatrix} 0 & -2 & 5 & 1 & -2 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{4}{9} & -\frac{2}{9} \end{bmatrix}$ 1st row − 5(3rd row), 2nd row + 3rd row

$\begin{bmatrix} 0 & -2 & 0 & -\frac{2}{3} & \frac{2}{9} & \frac{10}{9} \\ 1 & 0 & 0 & \frac{1}{3} & \frac{5}{9} & -\frac{2}{9} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{4}{9} & -\frac{2}{9} \end{bmatrix}$ Divide 1st row by −2. Rearrange rows.

$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} & \frac{5}{9} & -\frac{2}{9} \\ 0 & 1 & 0 & \frac{1}{3} & -\frac{1}{9} & -\frac{5}{9} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{4}{9} & -\frac{2}{9} \end{bmatrix}$

$$\begin{bmatrix} 0 & -2 & 3 \\ 1 & 0 & -1 \\ 1 & 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{5}{9} & -\frac{2}{9} \\ \frac{1}{3} & -\frac{1}{9} & -\frac{5}{9} \\ \frac{1}{3} & -\frac{4}{9} & -\frac{2}{9} \end{bmatrix}$$

25. $\begin{bmatrix} 2 & 5 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 12 & 11 \\ 5 & 18 \end{bmatrix}$

$\begin{bmatrix} 2 & 5 \\ -3 & 4 \end{bmatrix}^{-1}\begin{bmatrix} 2 & 5 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ -3 & 4 \end{bmatrix}^{-1}\begin{bmatrix} 12 & 11 \\ 5 & 18 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ -3 & 4 \end{bmatrix}^{-1}\begin{bmatrix} 12 & 11 \\ 5 & 18 \end{bmatrix}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ -3 & 4 \end{bmatrix}^{-1}\begin{bmatrix} 12 & 11 \\ 5 & 18 \end{bmatrix}$.

To find $\begin{bmatrix} 2 & 5 \\ -3 & 4 \end{bmatrix}^{-1}$: $\begin{bmatrix} 2 & 5 & 1 & 0 \\ -3 & 4 & 0 & 1 \end{bmatrix}$ becomes $\begin{bmatrix} 1 & 0 & \frac{4}{23} & \frac{-5}{23} \\ 0 & 1 & \frac{3}{23} & \frac{2}{23} \end{bmatrix}$,

so $\begin{bmatrix} 2 & 5 \\ -3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{4}{23} & \frac{-5}{23} \\ \frac{3}{23} & \frac{2}{23} \end{bmatrix}$.

Thus $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \frac{4}{23} & \frac{-5}{23} \\ \frac{3}{23} & \frac{2}{23} \end{bmatrix}\begin{bmatrix} 12 & 11 \\ 5 & 18 \end{bmatrix}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}$, so $a = 1, b = -2, c = 2, d = 3$.

Campfire queen Cycling champion Sentimental geologist*

Learn more about Marjon Walrod and tell us more about you. Visit pwc.com/bringit.

Your life. You can bring it with you.

*connectedthinking

PricewaterhouseCoopers

Exercise 11–1

Parabola $y = \frac{1}{4p}(x-h)^2 + k$ is the equation of a parabola with vertex at (h, k), focus $(h, k+p)$ and directrix the line $y = k-p$.

1. $y = -2x^2$ $\quad V(0, 0)$

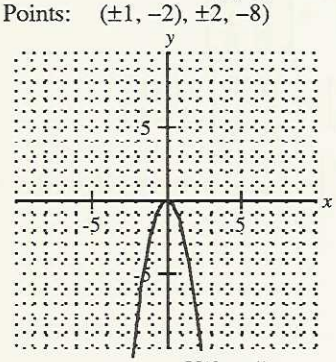
$\frac{1}{4p} = -2;$ $\quad p = -\frac{1}{8}$

focus: $(0, 0+(-\frac{1}{8}))$ $\quad (0, -\frac{1}{8})$

directrix: $\quad y = \frac{1}{8}$

Intercepts: $\quad (0, 0)$

Additional Points: $(\pm1, -2), \pm2, -8)$



5. $y = x^2 - 4$ $\quad V(0, -4)$

$\frac{1}{4p} = 1$ $\quad p = \frac{1}{4}$

focus: $(0, -4+\frac{1}{4})$ $\quad (0, -3\frac{3}{4})$

directrix: $y = -4 - \frac{1}{4}$ $\quad y = -4\frac{1}{4}$

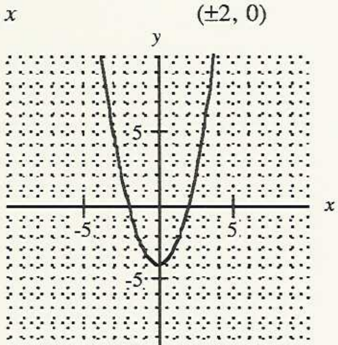
Intercepts:

$x=0$: $y = 0 - 4 = -4$ $\quad (0, -4)$

$y=0$: $0 = x^2 - 4$

$4 = x^2$

$\pm2 = x$ $\quad (\pm2, 0)$



9. $y = 2(x-3)^2$

The graph of $y = 2x^2$ shifted 3 units right. $V(3, 0)$

$\frac{1}{4p} = 2$; $\quad p = \frac{1}{8}$

focus: $(3, 0 + \frac{1}{8})$ $\quad (3, \frac{1}{8})$

directrix: $\quad y = -\frac{1}{8}$

Intercepts:

$x=0$: $y = 2(9) = 18$ $\quad (0, 18)$

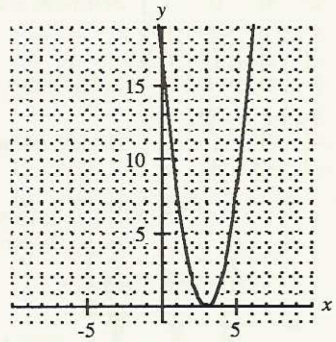
$y=0$: $0 = 2(x-3)^2$

$0 = (x-3)^2$

$0 = x - 3$

$x = 3$ $(3, 0)$

Additional Points: $(2, 2), (4, 2)$



13. $y = -(x+1)^2$

The graph of $y = x^2$ shifted left 1 unit and flipped vertically about the x–axis.

$V(-1, 0)$ $\quad \frac{1}{4p} = -1;$

$p = -\frac{1}{4}$

focus: $(-1, 0 - \frac{1}{4})$ $\quad (-1, -\frac{1}{4})$

directrix: $\quad y = 0 - (-\frac{1}{4})$

$y = \frac{1}{4}$

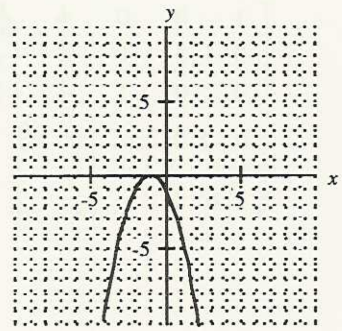
Intercepts:

$x=0$: $y = -1$ $\quad (0, -1)$

$y=0$: $0 = -(x+1)^2$

$x = -1$ $\quad (-1, 0)$

Additional Points: $(-2, -1)$



17. $y = -2(x+2)^2 + 1$ $\quad V(-2,1)$

$\frac{1}{4p} = -2$ $\quad p = -\frac{1}{8}$

focus: $(-2,1 - \frac{1}{8})$ $\quad (-2,\frac{7}{8})$

directrix: $y = 1 - (-\frac{1}{8})$ $\quad y = 1\frac{1}{8}$

Intercepts:

$x=0$: $y = -2(2)^2 + 1$ $\quad (0,-7)$

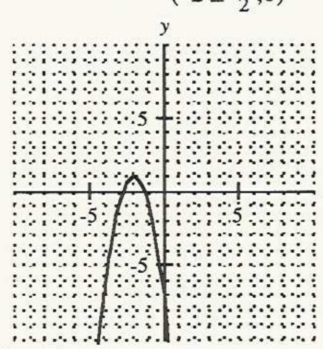
$y=0$: $0 = -2(x+2)^2 + 1$

$(x+2)^2 = \frac{1}{2}$

$x + 2 = \pm\frac{\sqrt{2}}{2}$

$x = -2 \pm \frac{\sqrt{2}}{2} \approx -1.3, -2.7$

$(-2 \pm \frac{\sqrt{2}}{2}, 0)$



21. $y = -x^2 + 6x - 7$

$y = -(x^2 - 6x) - 7$

$y = -(x^2 - 6x + 9) - 7 + 9$

$y = -(x-3)^2 + 2$ $\quad V(3, 2)$

$\frac{1}{4p} = -1$ $\quad p = -\frac{1}{4}$

focus: $(3, 2 - \frac{1}{4}) =$ $\quad (3, 1\frac{3}{4})$

directrix: $y = 2 - (-\frac{1}{4}) = 2\frac{1}{4}$ $\quad y = 2\frac{1}{4}$

Intercepts:

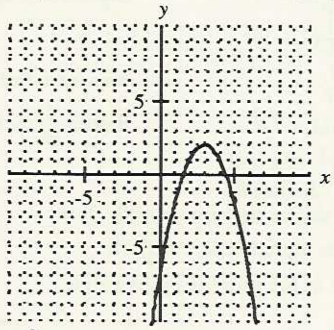
x=0: $y = -7$ $\quad (0, -7)$

y=0: $0 = -(x-3)^2 + 2$

$(x-3)^2 = 2$

$x - 3 = \pm\sqrt{2}$

$x = 3 \pm \sqrt{2} \approx 4.4,\ 1.6$ $\quad (1.6, 0), (4.4, 0)$



25. $y = 2x^2 - x - 3$

$y = 2(x^2 - \frac{1}{2}x) - 3$

$y = 2(x^2 - \frac{1}{2}x + \frac{1}{16}) - 3 - 2(\frac{1}{16})$

$y = 2(x - \frac{1}{4})^2 - 3\frac{1}{8}$ $\quad V(\frac{1}{4}, -3\frac{1}{8})$

$\frac{1}{4p} = 2$ $\quad p = \frac{1}{8}$

focus: $(\frac{1}{4}, -3\frac{1}{8} + \frac{1}{8})$ $\quad (\frac{1}{4}, -3)$

directrix: $y = -3\frac{1}{8} - \frac{1}{8} = -3\frac{1}{4}$ $\quad y = -3\frac{1}{4}$

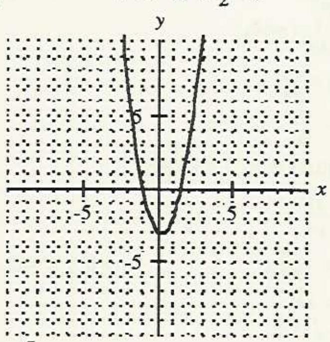
Intercepts:

x=0: $y = -3$ $\quad (0, -3)$

y=0: $0 = 2x^2 - x - 3$

$0 = (2x - 3)(x + 1)$

$x = \frac{3}{2}$ or -1 $\quad (-1, 0), (1\frac{1}{2}, 0)$



29. $y = x^2 - 3x - 5$

$y = x^2 - 3x + \frac{9}{4} - 5 - \frac{9}{4}$

$y = (x - 1\frac{1}{2})^2 - 7\frac{1}{4}$ $\quad V(1\frac{1}{2}, -7\frac{1}{4})$

$\frac{1}{4p} = 1$ $\quad p = \frac{1}{4}$

Focus: $(\frac{3}{2}, -\frac{29}{4} + \frac{1}{4})$ $\quad (1\frac{1}{2}, -7)$

Directrix: $y = -\frac{29}{4} - \frac{1}{4} = -7\frac{1}{2}$ $\quad y = -7\frac{1}{2}$

Intercepts:

x=0: $y = -5$ $\quad (0, -5)$

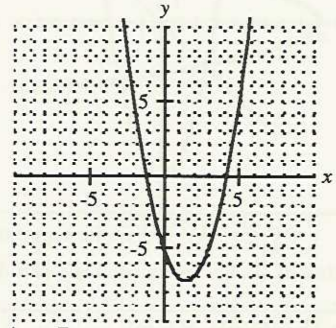
y=0: $0 = (x - 1\frac{1}{2})^2 - 7\frac{1}{4}$

$\frac{29}{4} = (x - \frac{3}{2})^2$

$\pm\sqrt{\frac{29}{4}} = x - \frac{3}{2}$

$x = \frac{3}{2} \pm \frac{\sqrt{29}}{2}$

$x = \frac{3 \pm \sqrt{29}}{2} \approx 4.2, -2.7$ $\quad (-2.7, 0), (4.2, 0)$



33. $y = -3x^2 - 4x + 7$

$y = -3(x^2 + \frac{4}{3}x) + 7$

$y = -3(x^2 + \frac{4}{3}x + \frac{4}{9}) + 7 + 3(\frac{4}{9})$

$y = -3(x + \frac{2}{3})^2 + 8\frac{1}{3}$ $\quad V(-\frac{2}{3}, 8\frac{1}{3})$

$\frac{1}{4p} = -3$ $\quad p = -\frac{1}{12}$

Focus: $(-\frac{2}{3}, \frac{25}{3} - \frac{1}{12})$ $\quad (-\frac{2}{3}, 8\frac{1}{4})$

Directrix: $y = \frac{25}{3} - (-\frac{1}{12}) = 8\frac{5}{12}$ $\quad y = 8\frac{5}{12}$

Intercepts:

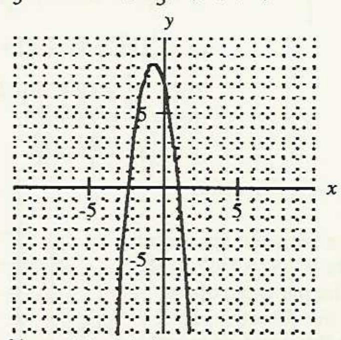
x=0: $y = 7$ $\quad (0, 7)$

y=0: $0 = -3x^2 - 4x + 7$

$0 = 3x^2 + 4x - 7$

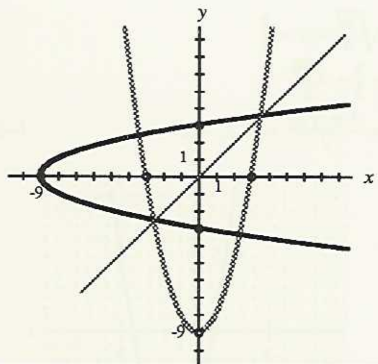
$0 = (3x + 7)(x - 1)$

$x = -\frac{7}{3}$ or 1 $\quad (-2\frac{1}{3}, 0), (1, 0)$



37. $x = y^2 - 9$; Since this relation expresses x as a function of y we first graph its inverse relation then reflect the graph about the line $y = x$. (ie: reverse the ordered pairs).

Inverse relation		Relation
$y = x^2 - 9$	$V(0, -9)$	$x = y^2 - 9$ $V(-9, 0)$
$\frac{1}{4p} = 1, p = \frac{1}{4}$		
Focus: $(0, -9 + \frac{1}{4})$	$(0, -8\frac{3}{4})$	$(-8\frac{3}{4}, 0)$
Directrix: $y = -9 - \frac{1}{4} = -9\frac{1}{4}$		$x = -9\frac{1}{4}$
Intercepts:		
x=0: $y = -9$	$(0, -9)$	$(-9, 0)$
y=0: $9 = x^2$		
$\pm 3 = x$	$(-3, 0)$ $(3, 0)$	$(0, -3)$ $(0, 3)$



Parabola $y = \frac{1}{4p}(x-h)^2 + k$ is the equation of a parabola with vertex at (h, k), focus $(h, k+p)$ and directrix the line $y = k-p$.

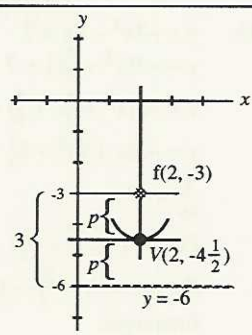
41. focus: $(2, -3)$, directix: $y = -6$

The distance from the focus to the directrix is $2|p|$. Thus

$2|p| = 3$

$|p| = \frac{3}{2}$.

The focus is above the directrix so the parabola opens up, and $p > 0$, so $p = \frac{3}{2}$. The vertex is halfway between the focus and directix, so it is at $(2, -4\frac{1}{2}) = (h, k)$. Thus $h = 2$, $k = -4\frac{1}{2}$.

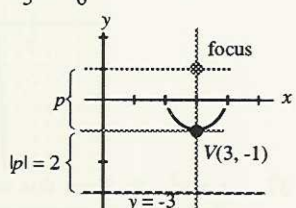


$$\frac{1}{4p} = \frac{1}{4(\frac{3}{2})} = \frac{1}{6}$$

$$y = \frac{1}{4p}(x-h)^2 + k$$
$$y = \frac{1}{6}(x-2)^2 + (-\frac{9}{2})$$
$$y = \frac{1}{6}(x^2 - 4x + 4) + \frac{9}{2}$$
$$6y = x^2 - 4x + 4 - 27$$
$$6y = x^2 - 4x - 23$$
$$y = \frac{1}{6}x^2 - \frac{2}{3}x - \frac{23}{6}$$

45. vertex: $(3, -1)$, directrix: $y = -3$

The distance between the vertex and directrix is $|p|$, which in this case is 2. The parabola opens away from the directrix, so it opens up in this case, so $p > 0$, so $p = +2$. We know $(h, k) = (3, -1)$.



$$y = \frac{1}{4p}(x-h)^2 + k$$
$$y = \frac{1}{4(2)}(x-3)^2 - 1$$
$$8y = (x^2 - 6x + 9) - 8$$
$$y = \frac{1}{8}x^2 - \frac{3}{4}x + \frac{1}{8}.$$

49. vertex: $(3, -1)$, x-intercepts: 2, 4

$$y = \frac{1}{4p}(x-h)^2 + k$$

[1] $y = \frac{1}{4p}(x-3)^2 - 1$ Replace h by 3, k by -1.

To find the value of p we can use the vact that we know the point $(x, y) = (2, 0)$ satisifies the equation (since it is one of the equation's x-intercepts). Thus we know that

$0 = \frac{1}{4p}(2-3)^2 - 1$ Replace x by 2, y by 0 in [1].

$1 = \frac{1}{4p}(2-3)^2$

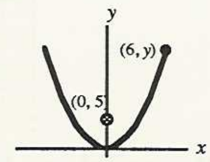
$1 = \frac{1}{4p}$. No need to find p itself.

$y = 1(x-3)^2 - 1$ Replace $\frac{1}{4p}$ by 1 in [1].

$y = x^2 - 6x + 8$

53. $h = 5$, $w = 12$; find d.

The figure shows that $p = 5$, $V(0, 0) = (h, k)$, so the equation is $y = \frac{1}{20}x^2$.



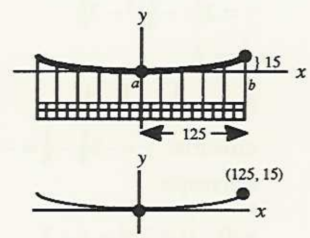
Since $d = y$ in the point $(6, y)$ we find y.

$y = \frac{1}{20}(6^2)$ Replace x by 6.

$y = \frac{36}{20} = \frac{9}{5}$.

Thus $d = y = \frac{9}{5}$.

57. The point on the cable at b is 15 feet higher than at a. Half the length of the bridge is 125 feet. This means that we could describe the parabola as shown. Thus $V(0, 0) = (h, k)$, so the equation is of the form $y = \frac{1}{4p}x^2$. We know that the point $(125, 15) = (x, y)$ satisfies this equation.



$$y = \frac{1}{4p}x^2$$
$$15 = \frac{1}{4p}(125^2)$$
$$\frac{15}{15625} = \frac{1}{4p}$$
$$\frac{3}{3125} = \frac{1}{4p}.$$

Thus the equation is $y = \frac{3}{3125}x^2$.

61. $y = -\frac{16}{v^2}x^2$

$-40 = -\frac{16}{8^2}x^2$ $v = 8$, $y = -40$.

$160 = x^2$

$x = \pm\sqrt{160}$

$x = 4\sqrt{10} \approx 12.6$ feet Assuming $x > 0$.

The horizontal distance travelled did double also, since $4\sqrt{10} = 2(2\sqrt{10})$.

Exercise 11–2

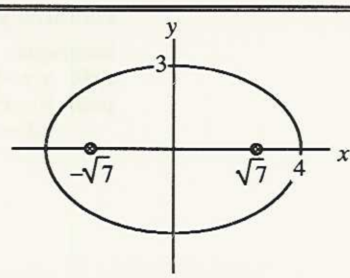
1. $\frac{x^2}{16} + \frac{y^2}{9} = 1$ Center: $(0, 0)$

$a = 4$, $b = 3$, $c = \sqrt{16-9} = \sqrt{7} \approx 2.6$

Major axis: x-axis

foci: $(\pm\sqrt{7}, 0) \approx (\pm 2.6, 0)$

Intercepts: $(\pm 4, 0)$, $(0, \pm 3)$



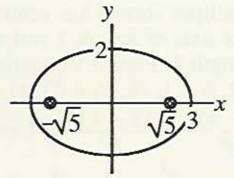
5. $\frac{x^2}{9} + \frac{y^2}{4} = 1$ Center: $(0, 0)$

$a = 3$, $b = 2$, $c = \sqrt{5} \approx 2.2$

Major axis: x-axis

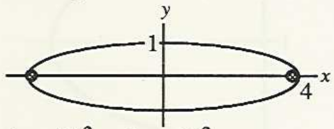
foci: $(\pm\sqrt{5}, 0)$

Intercepts: $(\pm 3, 0)$, $(0, \pm 2)$



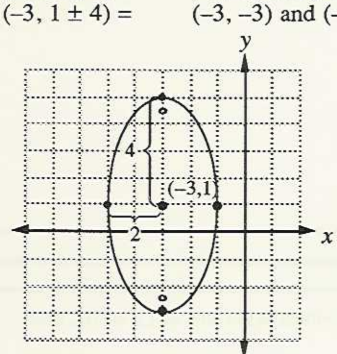
9. $\dfrac{x^2}{16} + y^2 = 1$ Center (0, 0)

$a = 4$, $b = 1$, $c = \sqrt{15} \approx 3.9$
Major axis: x-axis
foci: $(\pm\sqrt{15}, 0)$
Intercepts: $(\pm 4, 0)$, $(0, \pm 1)$



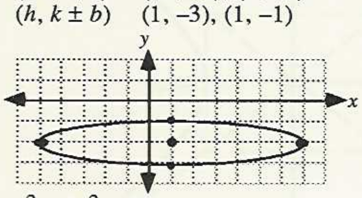
13. $\dfrac{(x+3)^2}{4} + \dfrac{(y-1)^2}{16} = 1$

Major axis parallel to y-axis.
$a = \sqrt{4} = 2$; $b = \sqrt{16} = 4$;
$c = \sqrt{16-4} = \sqrt{12}$
Center: $(h, k) =$ $(-3, 1)$
Foci: $(h, k \pm c) = (-3, 1 \pm \sqrt{12})$
$\approx (-3, -2.5)$,
$(-3, 4.5)$
End points of major/minor axes:
$(h \pm a, k)$, $(h, k \pm b)$
$(-3 \pm 2, 1) =$ $(-5, 1)$ and $(-1, 1)$
$(-3, 1 \pm 4) =$ $(-3, -3)$ and $(-3, 5)$



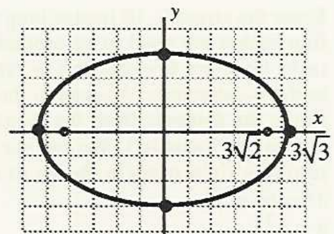
17. $\dfrac{(x-1)^2}{36} + (y+2)^2 = 1$

Center: (h, k) $(1, -2)$
$a = 6$, $b = 1$, $c = \sqrt{35} \approx 5.9$
Major axis parallel to the x-axis.
foci: $(h \pm c, k)$ $(1 \pm \sqrt{35}, -2)$
end points of major/minor axes:
$(h \pm a, k)$ $(-5, -2)$, $(7, -2)$
$(h, k \pm b)$ $(1, -3)$, $(1, -1)$



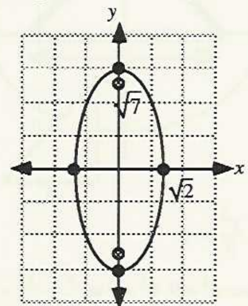
21. $x^2 + 3y^2 = 27$

$\dfrac{x^2}{27} + \dfrac{y^2}{9} = 1$ Center: (0, 0)
$a = 3\sqrt{3} \approx 5.2$, $b = 3$, $c = 3\sqrt{2} \approx 4.2$
Major axis parallel to the x-axis.
foci: $(\pm 3\sqrt{2}, 0)$
intercepts: $(\pm 3\sqrt{3}, 0)$, $(0, \pm 3)$



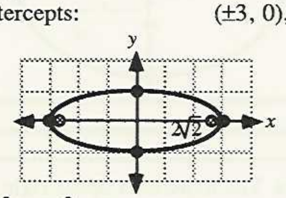
25. $9x^2 + 2y^2 = 18$

$\dfrac{x^2}{2} + \dfrac{y^2}{9} = 1$ Center: (0, 0)
$a = \sqrt{2}$, $b = 3$, $c = \sqrt{7} \approx 2.6$
Major axis parallel to the y-axis.
foci: $(0, \pm\sqrt{7})$
intercepts: $(\pm\sqrt{2}, 0)$, $(0, \pm 3)$



29. $x^2 + 9y^2 = 9$

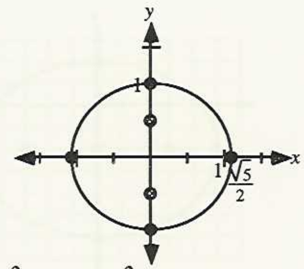
$\dfrac{x^2}{9} + y^2 = 1$ Center: (0, 0)
$a = 3$, $b = 1$, $c = 2\sqrt{2}$
Major axis parallel to the x-axis.
foci: $(\pm 2\sqrt{2}, 0)$
intercepts: $(\pm 3, 0)$, $(0, \pm 1)$



33. $4x^2 + 5y^2 = 5$

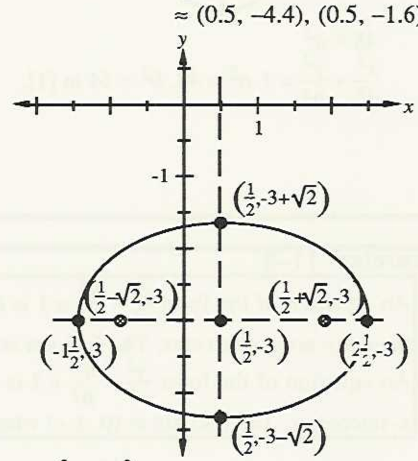
$\dfrac{4x^2}{5} + y^2 = 1$ Divide each term by 5.
$\dfrac{x^2}{\frac{5}{4}} + y^2 = 1$ $\frac{4}{5} = \dfrac{1}{\frac{5}{4}}$

$a = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$; $b = \sqrt{1} = 1$;
$c = \sqrt{\frac{5}{4} - 1} = \sqrt{\frac{1}{4}} = \frac{1}{2}$.
Center: (0, 0)
Major axis parallel to x-axis.
Foci: $(0 \pm c, 0) =$ $(\pm\frac{1}{2}, 0)$
Intercepts:
$x=0$: $5y^2 = 5$, $y = \pm 1$ $(0, \pm 1)$
$y=0$: $4x^2 = 5$, $x^2 = \frac{5}{4}$, $x = \pm\frac{\sqrt{5}}{2}$
$\approx (\pm 1.1, 0)$



37. $4x^2 - 4x + 8y^2 + 48y = -57$

$4(x^2 - x) + 8(y^2 + 6y) = -57$
$4(x^2 - x + \frac{1}{4}) + 8(y^2 + 6y + 9) =$
$-57 + 4(\frac{1}{4}) + 72$
$4(x - \frac{1}{2})^2 + 8(y+3)^2 = 16$
$\dfrac{(x - \frac{1}{2})^2}{4} + \dfrac{(y+3)^2}{2} = 1$
Center: $(h, k) =$ $(\frac{1}{2}, -3)$
$a = \sqrt{4} = 2$; $b = \sqrt{2}$; $c = \sqrt{4-2} = \sqrt{2}$
Major axis parallel to x-axis.
Foci: $(h \pm c, k) =$ $(\frac{1}{2} \pm \sqrt{2}, 0)$
End points of major/minor axes:
$(h \pm a, k)$ $(\frac{1}{2} \pm 2, -3)$
$= (-1\frac{1}{2}, -3)$ and $(2\frac{1}{2}, -3)$
$(h, k \pm b)$ $(\frac{1}{2}, -3 \pm \sqrt{2})$
$\approx (0.5, -4.4)$, $(0.5, -1.6)$

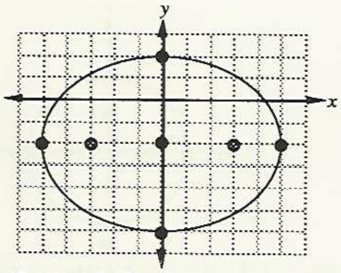


41. $x^2 + 2y^2 + 8 = 0$

$x^2 + 2y^2 = -8$
There is no real solution to this equation, so there is no graph for this relation.

45. $16x^2 + 25y^2 + 100y = 300$

$16x^2 + 25(y^2 + 4y + 4) = 300 + 25(4)$
$16x^2 + 25(y+2)^2 = 400$
$\dfrac{x^2}{25} + \dfrac{(y+2)^2}{16} = 1$
Center: $(h, k) =$ $(0, -2)$
$a = 5$, $b = 4$, $c = 3$
Major axis parallel to the x-axis.
foci: $(h \pm c, k)$ $(\pm 3, -2)$
end points of major/minor axes:
$(h \pm a, k)$ $(\pm 5, -2)$
$(h, k \pm b)$ $(0, -6)$, $(0, 2)$



49. foci: (0, 4) and (0, –4); one y–intercept at 8

[1] $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ The equation is of this form.

$b = 8$ y–intercept.

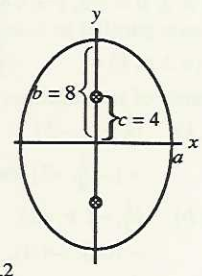
$c = 4$ Distance from center to a focus.

$b > a$ Major axis parallel to y-axis.

$c = \sqrt{b^2 - a^2}$

$4 = \sqrt{8^2 - a^2}$

$16 = 64 - a^2$



$48 = a^2$

$\frac{x^2}{48} + \frac{y^2}{64} = 1$ $a^2 = 48, b^2 = 64$ in [1].

53. Since the string is 10 inches long and four inches are used to go around the tacks there are 6 inches left to "stretch" to the y–intercepts. As seen in the figure the distance from the y–intercept at b to one focus is 3. We can use the resulting right triangle shown to find b.

$3^2 = b^2 + 2^2$

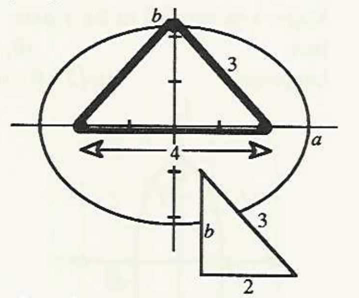
$5 = b^2$

$a > b$, so $c = \sqrt{a^2 - b^2}$

$2 = \sqrt{a^2 - 5}$

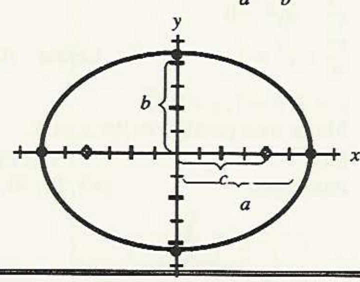
$4 = a^2 - 5$

$9 = a^2$



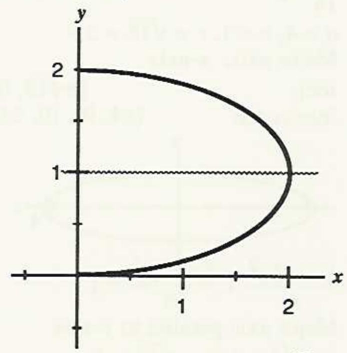
$\frac{x^2}{9} + \frac{y^2}{5} = 1$

Replace $a^2 = 9, b^2 = 5$ in $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



57. The ellipse shown has center at (0, 1), minor axis of length 2 and major axis of length 4. Find it's equation.

$a = 2, b = 1, (h, k) = (0, 1)$. Thus using $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ we obtain the equation $\frac{x^2}{4} + (y-1)^2 = 1$.



61. Problem 4 $\frac{c}{b} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$

65. Problem 8 $\frac{c}{b} = \frac{\sqrt{3}}{2}$

69. Problem 34 $\frac{c}{b} = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$

Exercise 11–3

An equation of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is a hyperbola. The hyperbola opens right and left. The x–intercepts are at $(\pm a, 0)$, and there are no y–intercepts. The foci are at $(\pm c, 0)$, where $c^2 = a^2 + b^2$.

An equation of the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ is a hyperbola which opens up and down. The y–intercepts are at $(0, \pm a)$, and there are no x–intercepts. The foci are at $(0, \pm c)$ where $c^2 = a^2 + b^2$.

1. $\frac{x^2}{25} - \frac{y^2}{16} = 1$

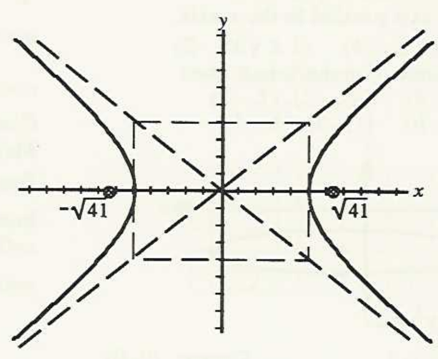
$c = \sqrt{25 + 16} = \sqrt{41}$

foci: $(\pm \sqrt{41}, 0)$

$a = 5$

$b = 4$

Major axis is horizontal.



5. $x^2 - \frac{y^2}{6} = 1$

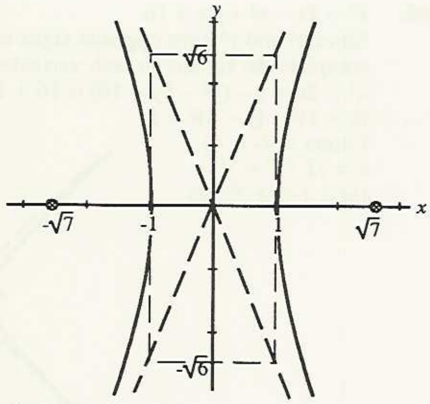
$c = \sqrt{6+1} = \sqrt{7}$

foci: $(\pm\sqrt{7}, 0)$

$a = 1$

$b = \sqrt{6}$

Major axis is horizontal.



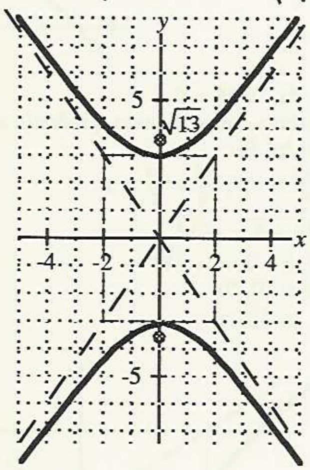
9. $\frac{y^2}{9} - \frac{x^2}{4} = 1$

$a = 3,\ b = 2,$

$c = \sqrt{13}$

Major axis is vertical.

foci: $(0, \pm\sqrt{13})$



13. $4y^2 - x^2 = 2$

$2y^2 - \frac{x^2}{2} = 1$

$\frac{y^2}{\frac{1}{2}} - \frac{x^2}{2} = 1$

$c = \sqrt{2 + \frac{1}{2}} = \sqrt{\frac{5}{2}} = \sqrt{\frac{10}{4}} = \frac{\sqrt{10}}{2}$

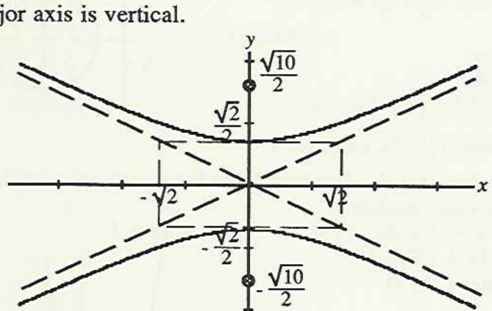
foci: $(0, \pm\frac{\sqrt{10}}{2})$

Intercepts:

$a = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$

$b = \sqrt{2}$

Major axis is vertical.



17. $16y^2 - x^2 = -16$

$-y^2 + \frac{x^2}{16} = 1$

$\frac{x^2}{16} - y^2 = 1$

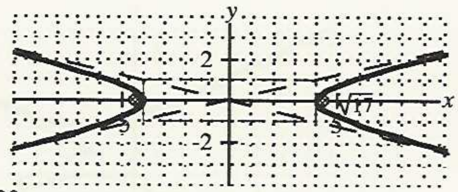
$a = 4,\ b = 1,$

$c = \sqrt{17}$

Major axis is horizontal.

foci:

$(\pm\sqrt{17}, 0)$



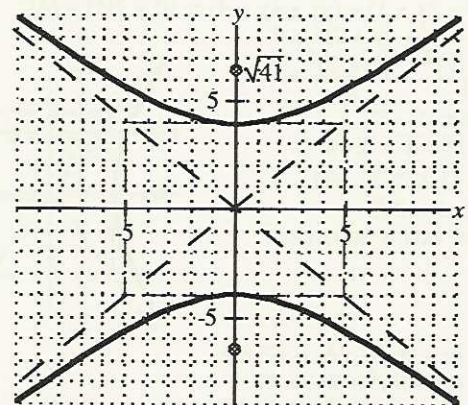
21. $25y^2 - 16x^2 = 400$

$\frac{y^2}{16} - \frac{x^2}{25} = 1$

$a = 4,\ b = 5,\ c = \sqrt{41}$

Major axis is vertical.

foci: $(0, \pm\sqrt{41})$



25. $\frac{(x-2)^2}{100} - \frac{(y+3)^2}{25} = 1$

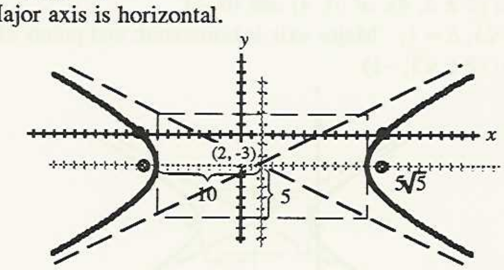
Center: $(2, -3)$

$c = \sqrt{100 + 25} = \sqrt{125} = 5\sqrt{5}$

foci: $(2 \pm 5\sqrt{5}, -3)$

$a = 10,\ b = 5$

Major axis is horizontal.



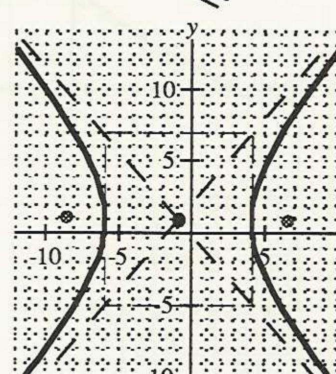
29. $\frac{(x+1)^2}{25} - \frac{(y-1)^2}{36} = 1$

$a = 5,\ b = 6,\ c = \sqrt{61}$

Center: $(-1, 1)$

Major axis is horizontal.

foci: $(-1 \pm \sqrt{61}, 1)$



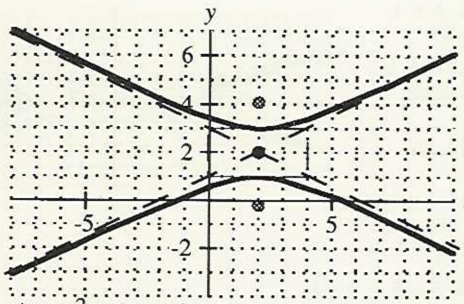
33. $(y-2)^2 - \frac{(x-2)^2}{4} = 1$

$a = 1,\ b = 2,\ c = \sqrt{5}$

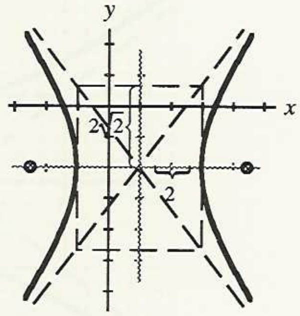
Center: $(2, 2)$

Major axis is vertical.

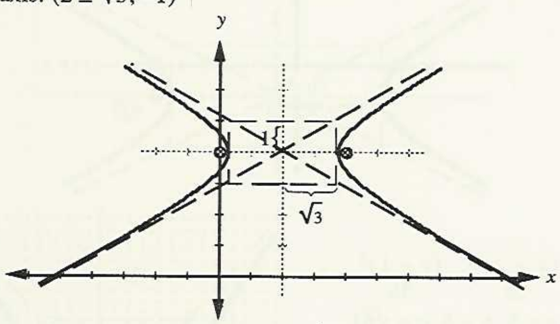
foci: $(2, 2 \pm \sqrt{5})$



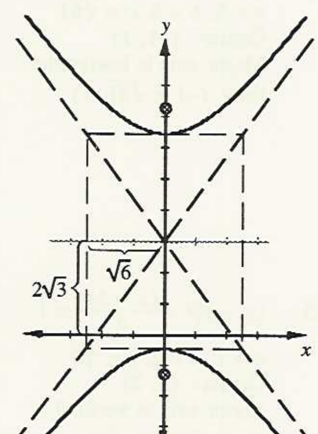
37. $2x^2 - 4x - y^2 - 4y - 10 = 0$
$2(x^2 - 2x) - (y^2 + 4y) = 10$
$2(x^2 - 2x + 1) - (y^2 + 4y + 4) = 10 + 2(1) - (4)$
$2(x-1)^2 - (y+2)^2 = 8$
$\dfrac{(x-1)^2}{4} - \dfrac{(y+2)^2}{8} = 1$
Center (1, –2)
$c = \sqrt{8+4} = 2\sqrt{3}$
foci: $(1 \pm 2\sqrt{3}, -2)$
$a = 2, b = 2\sqrt{2}$
Major axis is horizontal.



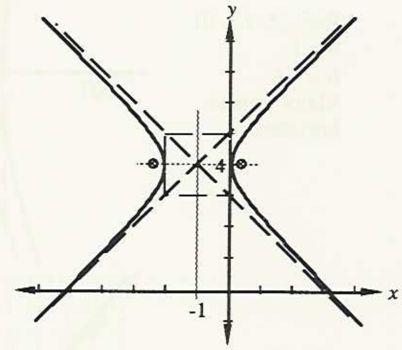
41. $x^2 - 4x - 3y^2 - 24y - 47 = 0$
$x^2 - 4x - 3(y^2 + 8y) = 47$
$x^2 - 4x + 4 - 3(y^2 + 8y + 16) = 47 + 4 - 3(16)$
$(x-2)^2 - 3(y+4)^2 = 3$
$\dfrac{(x-2)^2}{3} - (y-4)^2 = 1$
Center (2, 4)
$c = \sqrt{3+1} = 2$
foci: $(2 \pm 2, 4)$, or (4, 4) and (0, 4)
$a = \sqrt{3}, b = 1$; Major axis is horizontal; end points of major axis: $(2 \pm \sqrt{3}, -1)$



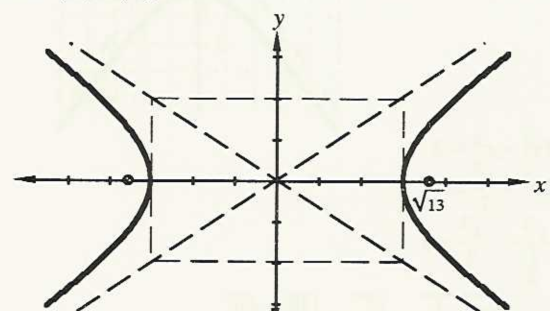
45. $2y^2 - 12y - 4x^2 = 6$
$2(y^2 - 6y) - 4x^2 = 6$
$2(y^2 - 6y + 9) - 4x^2 = 6 + 2(9)$
$2(y-3)^2 - 4x^2 = 24$
$\dfrac{(y-3)^2}{12} - \dfrac{x^2}{6} = 1$
Center (0, 3)
$c = \sqrt{12+6} = 3\sqrt{2}$
foci: $(0, 3 \pm 3\sqrt{2})$
$a = 2\sqrt{3}, b = \sqrt{6}$
Major axis is vertical.



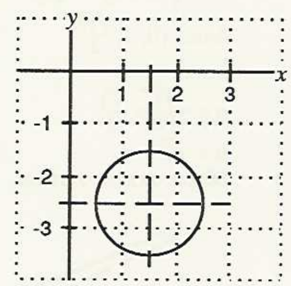
49. $x^2 + 2x - y^2 + 8y = 16$
Since x^2 and y^2 have opposite signs this is a hyperbola. We complete the square on both variables.
$x^2 + 2x + 1 - (y^2 - 8y + 16) = 16 + 1 - (16)$
$(x+1)^2 - (y-4)^2 = 1$
Center at (–1, 4)
$c = \sqrt{1+1} = \sqrt{2}$
foci: $(-1 \pm \sqrt{2}, 4)$



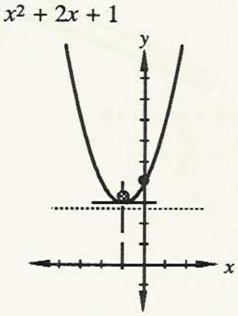
53. $9y^2 - 4x^2 + 36 = 0$
Hyperbola, since the two quadratic terms have opposite signs; divide by 36:
$\dfrac{y^2}{4} - \dfrac{x^2}{9} = -1$, $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$
$c = \sqrt{9+4} = \sqrt{13}$
$a = 3, b = 2, c = \sqrt{13}$
Center: (0, 0)
foci: $(\pm\sqrt{13}, 0)$



57. $4x^2 - 12x + 4y^2 + 20y + 30 = 0$
Circle since the two quadratic terms have the same coefficients.
$x^2 - 3x + y^2 + 5y = -\frac{15}{2}$
$x^2 - 3x + \frac{9}{4} + y^2 + 5y + \frac{25}{4}$
= –\f(15,2) +
$\frac{9}{4} + \frac{25}{4}$
$(x - \frac{3}{2})^2 + (y + \frac{5}{2})^2 = 1$
Center: $(\frac{3}{2}, -\frac{5}{2})$, radius = 1



61. $y = x^2 + 2x + 4$; Parabola since the equation is quadratic in only one variable.
$y - 4 = x^2 + 2x$, $y - 4 + 1 = x^2 + 2x + 1$
$y = (x+1)^2 + 3$
Center (–1, 3)
$\frac{1}{4p} = 1, p = \frac{1}{4}$;
focus: $(-1, 3 + \frac{1}{4}) = (-1, 3\frac{1}{4})$
directrix: $y = 3 - \frac{1}{4} = 2\frac{3}{4}$
Intercepts:
x=0: $y = 4$;
y=0: $-3 = (x+1)^2$; no real solution.



65. $4y = 4x^2 - 20x + 23$; Parabola since the equation is quadratic in only one variable.

$y - \frac{23}{4} = x^2 - 5x$

$y - \frac{23}{4} + \frac{25}{4} = x^2 - 5x + \frac{25}{4}$

$y = (x - \frac{5}{2})^2 + \frac{1}{2}$

Center: $(2\frac{1}{2}, -\frac{1}{2})$

$\frac{1}{4p} = 1, p = \frac{1}{4}$; focus:

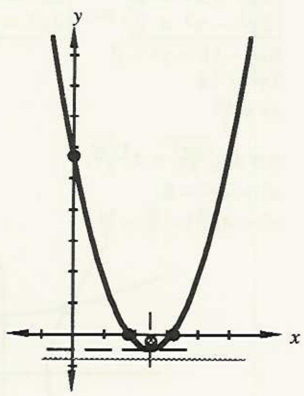
$(2\frac{1}{2}, -\frac{1}{2} + \frac{1}{4}) = (2\frac{1}{2}, -\frac{1}{4})$

Directrix: $y = -\frac{1}{2} - \frac{1}{4}$
$= -\frac{3}{4}$

Intercepts:

x=0: $4y = 23$, $y = \frac{23}{4}$

y=0: $-\frac{1}{2} = (x - \frac{5}{2})^2$; no real intercept.



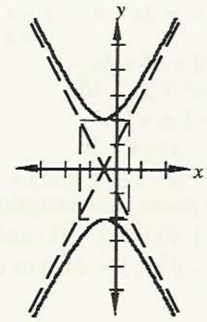
69. $y^2 - 4x^2 - 4x = 5$

$y^2 - 4(x^2 + x) = 5$

$y^2 - 4(x^2 + x + \frac{1}{4}) = 5 - 4(\frac{1}{4})$

$y^2 - 4(x + \frac{1}{2})^2 = 4$

$\frac{y^2}{4} - (x + \frac{1}{2})^2 = 1$



Exercise 11–4

1. $\boxed{\begin{array}{l} y = 2x + 1 \\ y = x^2 + x - 5 \end{array}}$

$2x + 1 = x^2 + x - 5$

$0 = x^2 - x - 6$

$0 = (x - 3)(x + 2)$

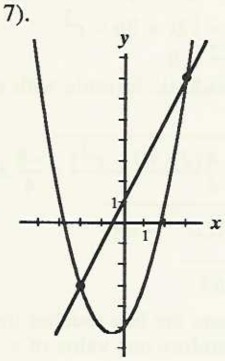
$x = -2$ or 3.

$y = 2x + 1$

$x = -2$: $y = 2(-2) + 1 = -3$

$x = 3$: $y = 2(3) + 1 = 7$

Thus the points of intersection are $(-2, -3)$ and $(3, 7)$.



5. $\boxed{\begin{array}{l} y = 3x^2 - 2x - 4 \\ y = x^2 + x + 1 \end{array}}$

$3x^2 - 2x - 4 = x^2 + x + 1$

$2x^2 - 3x - 5 = 0$

$(2x - 5)(x + 1) = 0$

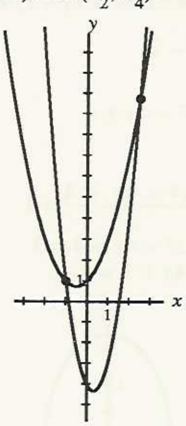
$x = \frac{5}{2}$ or -1

$y = x^2 + x + 1$

$x = -1$: $y = (-1)^2 + (-1) + 1 = 1.$

$x = \frac{5}{2}$: $y = (\frac{5}{2})^2 + \frac{5}{2} + 1 = \frac{25}{4} + \frac{10}{4} + \frac{4}{4}$
$= \frac{39}{4}.$

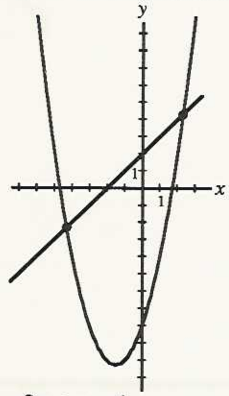
The points are $(-1, 1)$ and $(2\frac{1}{2}, 9\frac{3}{4})$.



9. $y = x^2 + 3x - 8 \qquad y - x = 2$
$y = x + 2$

$x + 2 = x^2 + 3x - 8$
$0 = x^2 + 2x - 10$
$x = -1 \pm \sqrt{11}$
$y = x + 2$
$y = (-1 \pm \sqrt{11}) + 2 = 1 \pm \sqrt{11}.$

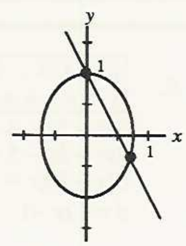
The points of intersection are $(-1 + \sqrt{11}, 1 + \sqrt{11})$ and $(-1 - \sqrt{11}, 1 - \sqrt{11})$ or about (2.3, 4.3) and (–4.3, –2.3).



13. $2x^2 + y^2 = 1; \qquad 2x + y = 1$
$y = -2x + 1$

[1] $y = -2x + 1$
$y^2 = 4x^2 - 4x + 1$

[2] $2x^2 + y^2 = 1$
$2x^2 + (4x^2 - 4x + 1) = 1$
$6x^2 - 4x = 0$
$2x(3x - 2) = 0$
$2x = 0$ or $3x - 2 = 0$
$x = 0$ or $\frac{2}{3}$



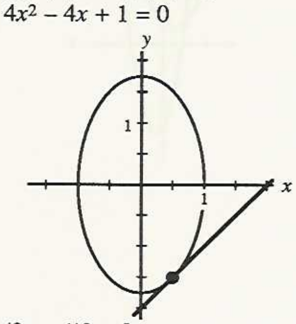
[1] $y = -2x + 1$
$x = 0$: $y = 0 + 1 = 1$
$x = \frac{2}{3}$: $y = -2(\frac{2}{3}) + \frac{3}{3} = -\frac{1}{3}$

The points of intersection are (0, 1) and $(\frac{2}{3}, -\frac{1}{3})$.

17. $x^2 + \frac{y^2}{3} = 1; \;\; y = x - 2$

[1] $y = x - 2$
$y^2 = x^2 - 4x + 4$

[2] $x^2 + \frac{y^2}{3} = 1$
$x^2 + \frac{x^2 - 4x + 4}{3} = 1$
$3x^2 + x^2 - 4x + 4 = 3$
$4x^2 - 4x + 1 = 0$

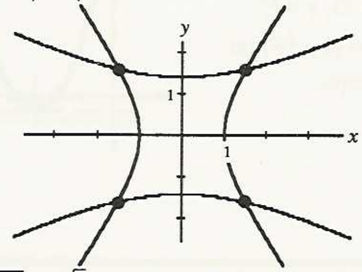


$(2x - 1)^2 = 0$
$2x - 1 = 0$
$x = \frac{1}{2}$

[1] $y = x - 2$
$x = \frac{1}{2}$: $y = \frac{1}{2} - \frac{4}{2} = -\frac{3}{2}$. There is one point of intersection, at $(\frac{1}{2}, -1\frac{1}{2})$.

21. $\boxed{\begin{array}{l} 4y^2 - x^2 = 8 \\ 2x^2 - y^2 = 2 \end{array}}$, so $\boxed{\begin{array}{l} 2x^2 = 8y^2 - 16 \\ 2x^2 = y^2 + 2 \end{array}}$

$8y^2 - 16 = y^2 + 2$
$7y^2 = 18$
$y^2 = \frac{18}{7}$
$y = \pm\sqrt{\frac{18 \cdot 7}{7 \cdot 7}} = \pm\frac{3\sqrt{14}}{7}.$
$x^2 = 4y^2 - 8$
$x^2 = 4(\frac{18}{7}) - \frac{56}{7} = \frac{16}{7}$



$x = \pm\sqrt{\frac{16 \cdot 7}{7 \cdot 7}} = \pm\frac{4\sqrt{7}}{7}$. Thus the points of intersection are $(\frac{4\sqrt{7}}{7}, \frac{3\sqrt{14}}{7})$, $(\frac{4\sqrt{7}}{7}, -\frac{3\sqrt{14}}{7})$, $(-\frac{4\sqrt{7}}{7}, \frac{3\sqrt{14}}{7})$, $(-\frac{4\sqrt{7}}{7}, -\frac{3\sqrt{14}}{7})$
$\approx$ (1.5, ±1.6), (–1.5, ±1.6) .

25. The circle has equation
$(x - 2)^2 + (y - 5)^2 = r^2.$

The circle touches the line $y = -x - 1$ at one point (since it is tangent to it); at this point, y may be replaced by $-x - 1$:

$(x - 2)^2 + (y - 5)^2 = r^2$
$(x - 2)^2 + ((-x - 1) - 5)^2 = r^2$
$x^2 - 4x + 4 + (-x - 6)^2 = r^2$
$x^2 - 4x + 4 + x^2 + 12x + 36 = r^2$
$2x^2 + 8x + 40 - r^2 = 0$

Now apply the quadratic formula with $a = 2$, $b = 8$, $c = 40 - r^2$:

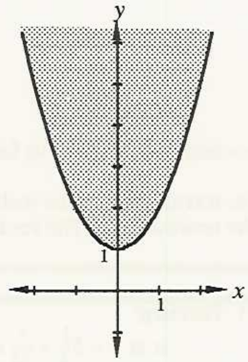
$$x = \frac{-8 \pm \sqrt{64 - 4(2)(40 - r^2)}}{4} = \frac{-8}{4} \pm \frac{\sqrt{8r^2 - 256}}{4}$$
$$= -2 \pm \frac{1}{4}\sqrt{4(2r^2 - 64)}$$
$$= -2 \pm \tfrac{1}{2}\sqrt{2r^2 - 64}.$$

We know that where the line touches the circle there is only one point, and therefore one value of x. This happens only if $2r^2 - 64$ is zero.

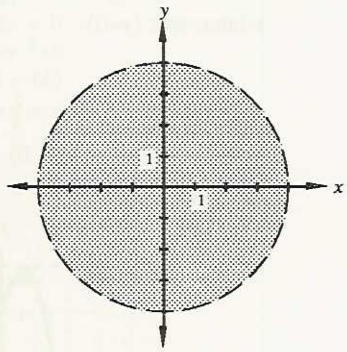
$2r^2 - 64 = 0$
$2r^2 = 64$
$r^2 = 32$

Thus we learn the value of r^2, and so the equation of the circle is
$(x - 2)^2 + (y - 5)^2 = 32.$

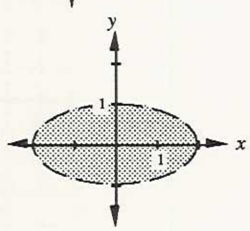
29. $y \geq x^2 + 1$
Graph the parabola $y = x^2 + 1$; use (0, 0) as a test point.
$y > x^2 + 1$
$0 > 0 + 1$
$0 > 1$
FALSE, so the solution is the part of the plane which does not contain the origin.



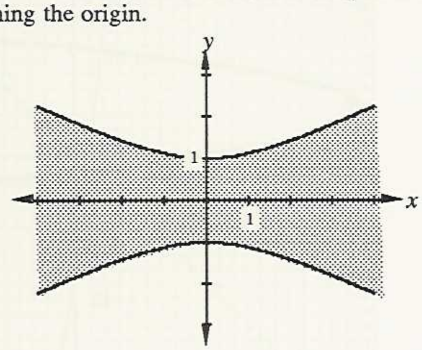
33. $x^2 + y^2 < 16$
Graph the circle $x^2 + y^2 = 16$; use (0, 0) as a test point.
$x^2 + y^2 < 16$
$0 + 0 < 16$
TRUE, so the solution is the part of the plane which contains the origin.



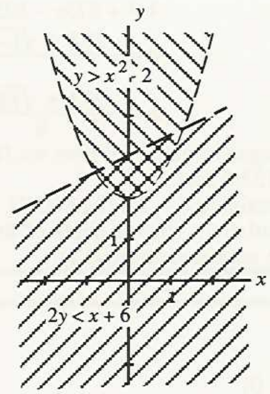
37. $\frac{x^2}{4} + y^2 < 1$
Graph the ellipse $\frac{x^2}{4} + y^2 = 1$; use (0, 0) as a test point.
$\frac{x^2}{4} + y^2 < 1$
$\frac{0}{4} + 0 < 1$
$0 < 1$ TRUE, so the solution contains the origin.



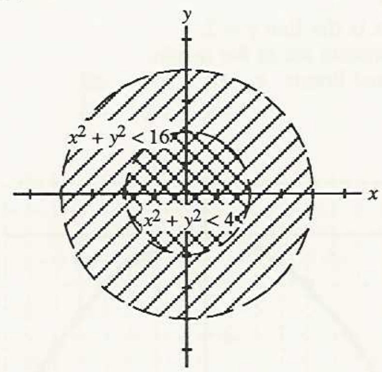
41. $y^2 - \frac{x^2}{4} \leq 1$
Graph the hyperbola $y^2 - \frac{x^2}{4} = 1$; use (0, 0) as a test point.
$y^2 - \frac{x^2}{4} < 1$
$0 - \frac{0}{4} < 1$
$0 < 1$ TRUE, so the solution is the part of the plane containing the origin.



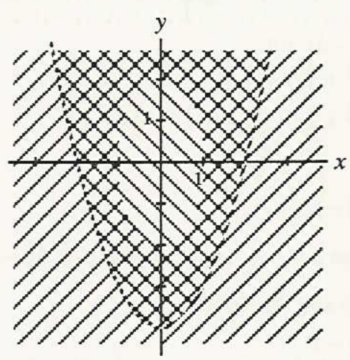
45. $y > x^2 + 2$
$2y < x + 6$



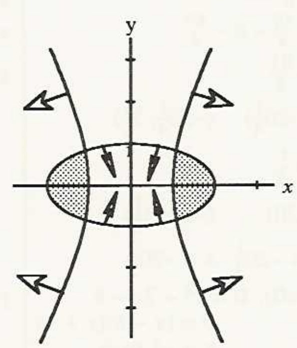
49. $x^2 + y^2 < 16$
$x^2 + y^2 < 4$



53. $x^2 + \frac{y^2}{4} > 1$
$y > x^2 - 4$



57. $x^2 - \frac{y^2}{4} > 1$
$\frac{x^2}{4} + y^2 < 1$



61. Since z is the time it takes to fall to the bottom of the well we know that $s = 16z^2$. Since the time to come back up is $3 - z$ seconds we know that $s = 1100(3 - z)$. Thus
$s = 16z^2$
$s = 1100(3 - z)$ so $16z^2 = 1100(3 - z)$.
$16z^2 + 1100z - 3300 = 0$

$$4z^2 + 275z - 825 = 0$$
$$z = \frac{-275 \pm \sqrt{(-275)^2 - 4(4)(-825)}}{2(4)}$$
$$z = \frac{-275 \pm \sqrt{88825}}{8} \approx -71.6,\ 2.879$$

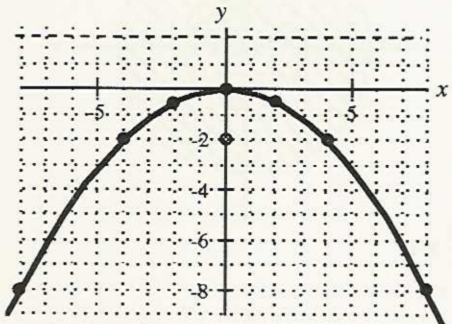
Ignoring the negative value for time we find that it takes 2.879 seconds for the rock to fall. At this point s is computed as $s = 16(2.879^2) \approx 132.6$ feet.
It takes the remaining $3 - 2.879$ or 0.121 seconds for the sound to travel back up the well, so $s = 1100(0.121) = 133.1$ feet.
Thus both calculations show a depth of the well of 133 feet, to the nearest foot. (The results will be the same if more decimal places are used in the approximation of z).

Chapter 11 Review

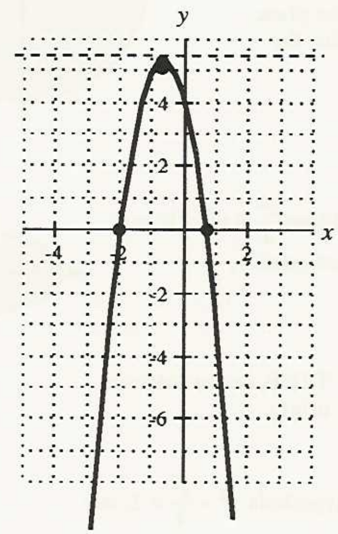
1. $y = -\frac{1}{8}x^2$
   Vertex at (0, 0).
   $\frac{1}{4p} = -\frac{1}{8}$, so $p = -2$. Thus the focus is at (–2, 0) and the directrix is the line $y = 2$.
   All intercepts are at the origin.

   Additional Points

x	± 8	± 4	± 2
y	-8	-2	$-\frac{1}{2}$

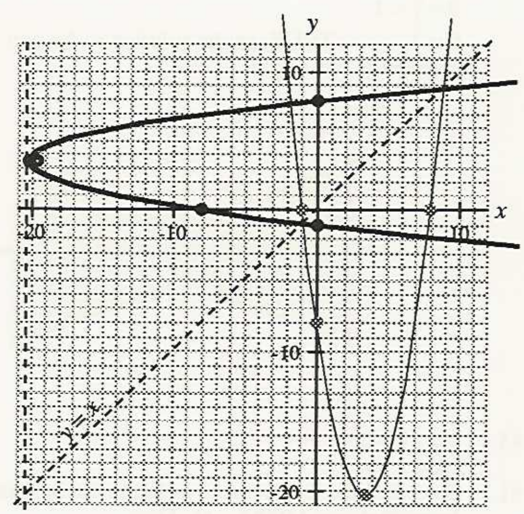


3. $y = -3x^2 - 4x + 4$
   $y = -3(x^2 + \frac{4}{3}x) + 4$
   $y = -3(x^2 + \frac{4}{3}x + \frac{4}{9}) + 4 + 3 \cdot \frac{4}{9}$
   $y = -3(x + \frac{2}{3})^2 + \frac{16}{3}$
   Vertex: $(-\frac{2}{3}, 5\frac{1}{3})$
   $\frac{1}{4p} = -3$, so $p = -\frac{1}{12}$, so focus is at $(-\frac{2}{3}, 5\frac{1}{3} - \frac{1}{12})$
   $= (-\frac{2}{3}, 5\frac{1}{4})$ and the directrix
   is at $y = 5\frac{1}{3} + \frac{1}{12}$ or $y = 5\frac{5}{12}$.
   x-intercepts (y=0): $0 = -3x^2 - 4x + 4$
   $3x^2 + 4x - 4 = 0$
   $(3x - 2)(x + 2) = 0$
   $x = \frac{2}{3}$ or -2
   $(\frac{2}{3}, 0)$, (–2, 0)
   y-intercept (x=0): $y = 4$ (0, 4)

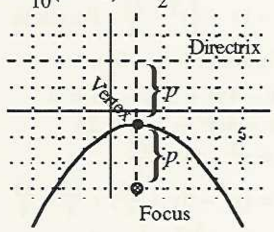


5. $x = y^2 - 7y - 8$
   We will graph the parabola $y = x^2 - 7x - 8$, then reflect the graph about the line $y = x$ to obtain the required graph. This means we reverse all ordered pairs for the final graph.

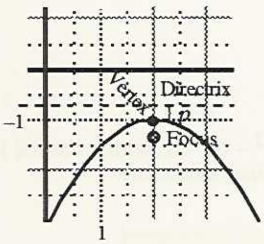
$y = x^2 - 7x - 8$	$x = y^2 - 7y - 8$
$y = x^2 - 7x + \frac{49}{4} - 8 - \frac{49}{4}$	$x = y^2 - 7y + \frac{49}{4} - 8 - \frac{49}{4}$
$y = (x - \frac{7}{2})^2 - \frac{81}{4}$	$x = (y - \frac{7}{2})^2 - \frac{81}{4}$
Vertex: $(3\frac{1}{2}, -20\frac{1}{4})$	$(-20\frac{1}{4}, 3\frac{1}{2})$
$\frac{1}{4p} = 1$, so $p = \frac{1}{4}$.	
Focus: $(3\frac{1}{2}, -20)$	$(-20, 3\frac{1}{2})$
Directrix: $y = -20\frac{1}{2}$	$x = -20\frac{1}{2}$
x-intercept (y=0): $0 = x^2 - 7x - 8$; $0 = (x - 8)(x + 1)$; $x = -1$ or 8; (–1, 0), (8, 0)	y-intercept: (0, –1), (0, 8)
y-intercept (x=0): (0, –8)	x-intercept: (–8, 0)



7. focus: (1, −3), directrix: $y = 2$
$p = -2\frac{1}{2}$, so $4p = -10$. Vertex $(1, -\frac{1}{2})$.
$$y = \frac{1}{4p}(x - h)^2 + k$$
$$y = -\frac{1}{10}(x - 1)^2 - \frac{1}{2}$$

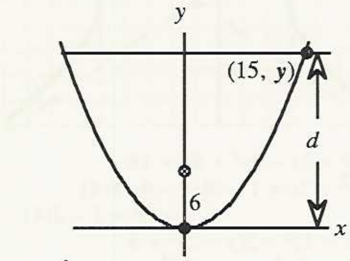


9. vertex: (2, −1), directrix: $y = -\frac{3}{4}$
$p = -\frac{1}{4}$, so $4p = -1$.
$$y = \frac{1}{4p}(x - h)^2 + k$$
$$y = -(x - 2)^2 - 1$$

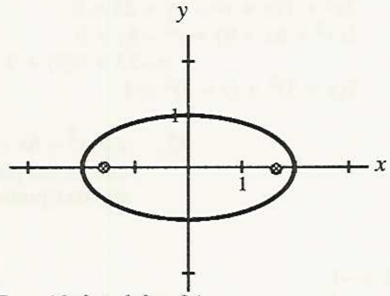


11. vertex: (3, −1), x-intercepts: $2\frac{1}{2}$, $3\frac{1}{2}$
It is not possible to deduce the value of p from this data, but we do know the vertex.
$$y = \frac{1}{4p}(x - h)^2 + k$$
$$y = \frac{1}{4p}(x - 3)^2 - 1$$
We can find p by using either of the x-intercepts for the value of a point (x,y) which lies on the parabola, and therefore which satisfies the equation. We will use the point $(2\frac{1}{2}, 0)$:
$$0 = \frac{1}{4p}(2\tfrac{1}{2} - 3)^2 - 1$$
$$1 = \frac{1}{4p}(-\tfrac{1}{2})^2$$
$$p = \tfrac{1}{4}(\tfrac{1}{4})$$
$$p = \frac{1}{16}, \text{ so } 4p = \frac{1}{4}, \text{ and } \frac{1}{4p} = 4.$$
Thus the equation is
$y = 4(x - 3)^2 - 1$.
We assume the vertex is at (0, 0) in each case.

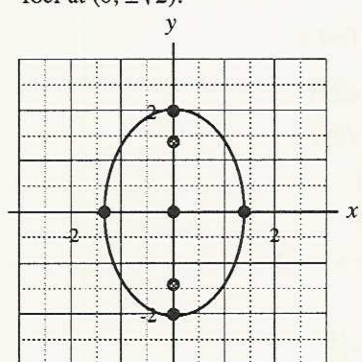
13. $h = 6$, $w = 30$; find d.
$p = 6$, so the equation is $y = \frac{1}{24}x^2$.
Let $x = 15$: $y = \frac{1}{24}(15^2) = \frac{225}{24} = 9\frac{3}{8}$, so $d = y = 9\frac{3}{8}$.



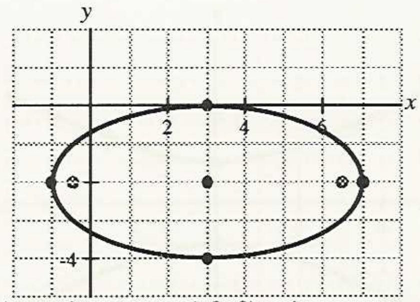
15. $\dfrac{x^2}{4} + y^2 = 1$
$a = 2, b = 1, c = \sqrt{4 - 1} = \sqrt{3} \approx 1.7$.
Center (0, 0), foci at $(-\sqrt{3},0)$, $(\sqrt{3},0)$.



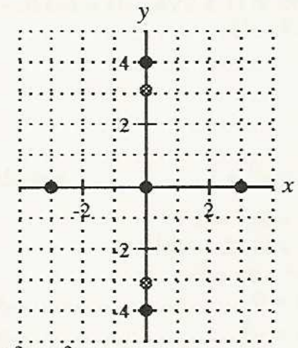
17. $12x^2 + 6y^2 = 24$
$\dfrac{x^2}{2} + \dfrac{y^2}{4} = 1$
$a = \sqrt{2} \approx 1.4$, $b = 2$,
$c = \sqrt{4 - 2} = \sqrt{2}$.
Center at (0, 0), major axis is vertical: foci at $(0, \pm\sqrt{2})$.



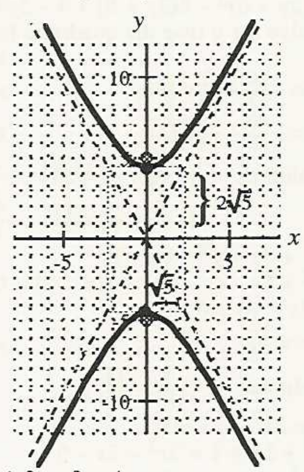
19. $x^2 - 6x + 4y^2 + 16y + 9 = 0$
$x^2 - 6x + 9 + 4(y^2 + 4y) = 0$
$(x - 3)^2 + 4(y^2 + 4y + 4) = 4(4)$
$(x - 3)^2 + 4(y + 2)^2 = 16$
$\dfrac{(x - 3)^2}{16} + \dfrac{(y + 2)^2}{4} = 1$
$a = 4, b = 2, c = \sqrt{12} = 2\sqrt{3} \approx 3.46$
Center at (3, −2).
Major axis is horizontal, foci at $(3 \pm 2\sqrt{3}, -2) \approx (-0.46, -2)$ and $(6.46, -2)$.



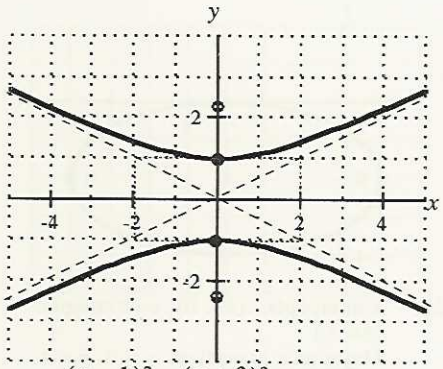
21. x-intercepts: (±3, 0); y-intercepts: (±4, 0)
Major axis is vertical. $a = 3$, $b = 4$, center at (0, 0).
$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$
$\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$



23. $\dfrac{y^2}{20} - \dfrac{x^2}{5} = 1$
Center at (0, 0).
$a = \sqrt{20} = 2\sqrt{5} \approx 4.47$, $b = \sqrt{5} \approx 2.2$,
$c = \sqrt{25} = 5$
Foci at (0, ±5).



25. $4y^2 - x^2 = 4$
$y^2 - \dfrac{x^2}{4} = 4$
$a = 1, b = 2, c = \sqrt{5} \approx 2.2$
Foci at $(0, \pm\sqrt{5})$.

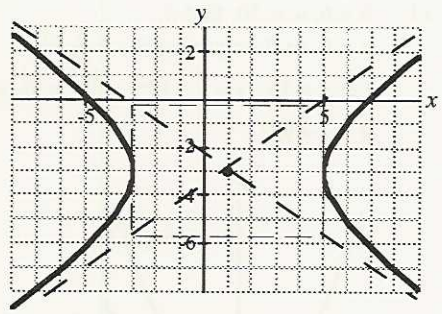


27. $\dfrac{(x-1)^2}{16} - \dfrac{(y+3)^2}{8} = 1$

Center at $(1, -3)$

$a = 4$, $b = \sqrt{8} = 2\sqrt{2} \approx 2.8$, $c = \sqrt{24} \approx 2\sqrt{6} \approx 4.9$

Foci at $(1 \pm 2\sqrt{6}, -3) \approx (-3.9, -3)$, $(5.9, -3)$.



29. $x^2 + 2x - 2y^2 + 8y = 16$

$x^2 + 2x + 1 - 2(y^2 - 4y + 4)$
$= 16 + 1 - 2(4)$

$(x+1)^2 - 2(y-2)^2 = 9$

$\dfrac{(x+1)^2}{9} - \dfrac{(y-2)^2}{\frac{9}{2}} = 1$

Hyperbola.

Center at $(-1, 2)$.

31. $2x^2 + 12x + y^2 - 6y + 23 = 0$

$2(x^2 + 6x + 9) + y^2 - 6y + 9$
$= -23 + 2(9) + 9$

$2(x+3)^2 + (y-3)^2 = 4$

$\dfrac{(x+3)^2}{2} + \dfrac{(y-3)^2}{4} = 1$ Ellipse.

Center at $(-3, 3)$.

33. $y = x^2 + 3x - 4$

$y = x^2 + 3x + \frac{9}{4} - 4 - \frac{9}{4}$

$y = (x + \frac{3}{2})^2 - \frac{25}{4}$ Parabola

Vertex at $(-\frac{1}{2}, -6\frac{1}{4})$.

35. $4x - 4y^2 + 20y - 23 = 0$

$4x = 4(y^2 - 5y) + 23$

$x = y^2 - 5y + \frac{23}{4}$

$x = (y^2 - 5y + \frac{25}{4}) + \frac{23}{4} - \frac{25}{4}$

$x = (y - \frac{5}{2})^2 - \frac{1}{2}$ Parabola (on its side).

Vertex at $(-\frac{1}{2}, -2\frac{1}{2})$.

37. $\dfrac{x^2}{2} + y^2 = 1$ $\qquad y = -2x - 1$

$x^2 + 2y^2 = 2$

$x^2 + 2(-2x-1)^2 = 2$

$9x^2 + 8x = 0$

$x = 0$ .... $y = -2(0) - 1 = -1$

$x = -\frac{8}{9}$ .... $y = -2(-\frac{8}{9}) - 1 = \frac{7}{9}$

Solution $(0, -1)$ and $(-\frac{8}{9}, \frac{7}{9})$

39. $\dfrac{(x-1)^2}{12} - \dfrac{y^2}{16} = 1$ $\qquad x - 2y = 6$

$4(x-1)^2 - 3y^2 = 48$ $\qquad x = 2y + 6$

$4x^2 - 8x + 4 - 3y^2 = 48$

$4(2y+6)^2 - 8(2y+6) + 4 - 3y^2 = 48$

Solve for y (use the quadratic formula) then find x.

$x = 2y + 6$

$y = -\frac{40}{13} + \frac{2}{13}\sqrt{231}$; $\qquad x = -\frac{2}{13} + \frac{4}{13}\sqrt{231}$

$y = -\frac{40}{13} - \frac{2}{13}\sqrt{231}$; $\qquad x = -\frac{2}{13} - \frac{4}{13}\sqrt{231}$

Solution: $(-\frac{2}{13} + \frac{4}{13}\sqrt{231}, -\frac{40}{13} + \frac{2}{13}\sqrt{231})$

$(-\frac{2}{13} - \frac{4}{13}\sqrt{231}, -\frac{40}{13} - \frac{2}{13}\sqrt{231})$

41. $x^2 + 5x - y^2 + 6y = 0$ $\qquad y = x - 2$

$x^2 + 5x - (x-2)^2 + 6(x-2) = 0$

$15x - 16 = 0$

$x = \frac{16}{15}$ .... $y = \frac{16}{15} - 2 = -\frac{14}{15}$

Solution $(1\frac{1}{15}, -\frac{14}{15})$

43. $y = 3x^2 - 2x - 5$ $\qquad y = x^2 + 2x + 1$

$x^2 + 2x + 1 = 3x^2 - 2x - 5$

$2x^2 - 4x - 6 = 0$

$x^2 - 2x - 3 = 0$

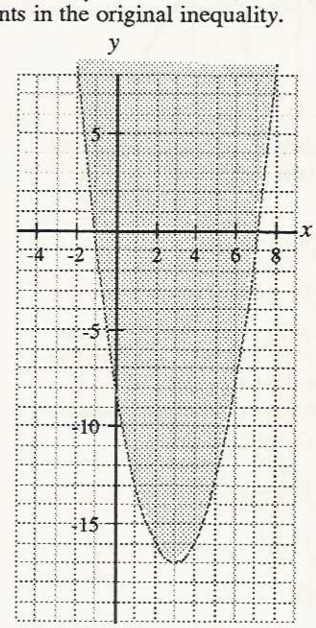
$x = 3$ .... $y = 3^2 + 2(3) + 1 = 16$

$x = -1$ .... $y = (-1)^2 + 2(-1) + 1 = 0$

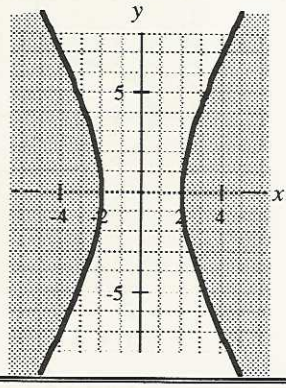
Solution $(3, 16)$ and $(-1, 0)$

45. $y > x^2 - 6x - 8$

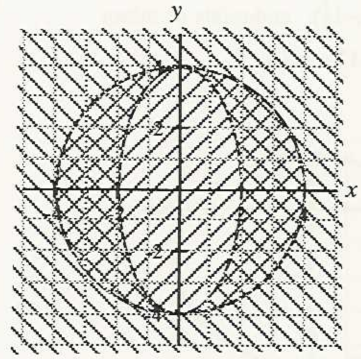
Graph the parabola $y = x^2 - 6x - 8$, with a dashed line, and use test points in the original inequality.



47. $4x^2 - y^2 \geq 16$
Graph the hyperbola $4x^2 - y^2 = 16$ (with a solid line) and use test points in the original inequality to determine the solution set.

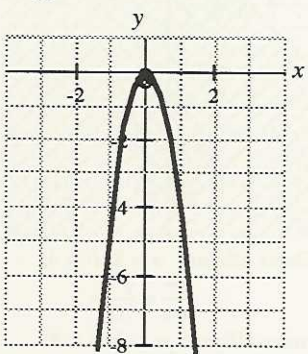


49. $x^2 + y^2 < 16$
$\frac{x^2}{4} + y^2 > 1$

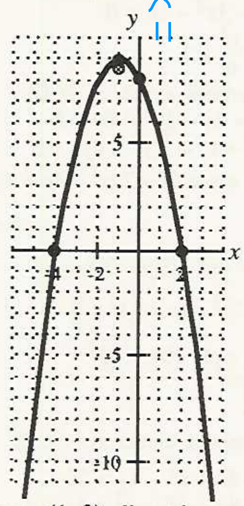


Chapter ~~7~~ 11 Test

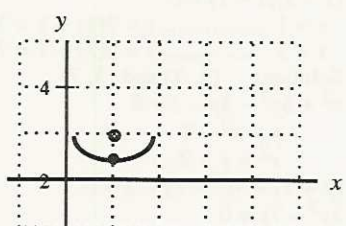
1. $y = -4x^2$
All intercepts and vertex at the origin.
$\frac{1}{4p} = -4$, so $p = -\frac{1}{16}$, so the focus is at $(0, -\frac{1}{16})$ and the directrix is the line $y = \frac{1}{16}$.



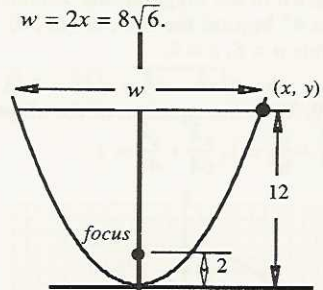
3. $y = -x^2 - 2x + 8$
$y = -(x^2 + 2x) + 8$
$y = -(x^2 + 2x + 1) + 8 + 1$
$y = -(x + 1)^2 + 9$
Vertex: $(-1, 9)$
x-intercept: $0 = -x^2 - 2x + 8$
$0 = x^2 + 2x - 8$
$0 = (x + 4)(x - 2)$
$x = -4$ or 2
$(-4, 0), (2, 0)$
y-intercept: $y = 8$ $\quad (0, 8)$
$\frac{1}{4p} = -1$, so $p = -\frac{1}{4}$.
Focus: $(-1, 8\frac{3}{4})$
Directrix: $y = 9\frac{1}{4}$



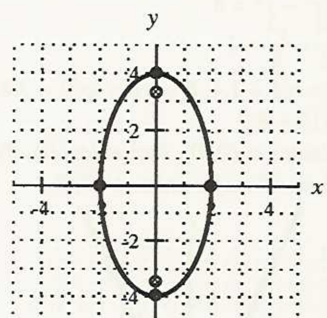
5. focus: $(1, 3)$, directrix: $y = 2$
Vertex is at $(1, 2\frac{1}{2})$. $p = \frac{1}{2}$, so $4p = 2$, so $\frac{1}{4p} = \frac{1}{2}$.
$y = \frac{1}{4p}(x - h)^2 - k$
$y = \frac{1}{2}(x - 1)^2 + 2\frac{1}{2}$



7. $h = 2$, $d = 12$; find w.
$p = 2$, $y = 12$; find x.
$y = \frac{1}{8}x^2$
$12 = \frac{1}{8}x^2$
$96 = x^2$
$x = \sqrt{96} = 4\sqrt{6}$.
$w = 2x = 8\sqrt{6}$.



9. $4x^2 + y^2 = 16$
$\frac{x^2}{4} + \frac{y^2}{16} = 1$
Center: $(0, 0)$, $a = 2$, $b = 4$, $c = \sqrt{12} = 2\sqrt{3} \approx 3.46$.
foci at $(0, \pm 2\sqrt{3})$



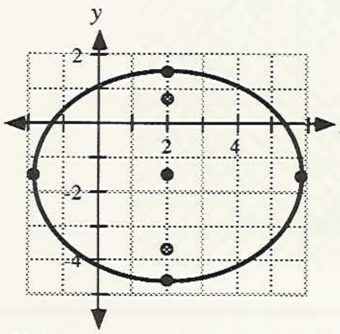
11. $2x^2 - 4x + 3y^2 + 9y = 15\frac{1}{4}$
$2(x^2 - 4x) + 3(y^2 + 3y) = 15\frac{1}{4}$
$2(x^2 - 4x + 4) + 3(y^2 + 3y + \frac{9}{4})$
$= 15\frac{1}{4} + 2(4) + 3(\frac{9}{4})$
$2(x - 2)^2 + 3(y + \frac{3}{2})^2 = 30$
$\frac{(x - 2)^2}{15} + \frac{(y + \frac{3}{2})^2}{10} = 1$
$a = \sqrt{15} \approx 3.9$, $b = \sqrt{10} \approx 3.2$,
$c = \sqrt{5} \approx 2.2$
Center $(2, -1\frac{1}{2})$, Foci at

$(2 \pm \sqrt{5}, -1\frac{1}{2}) \approx (-0.2, -1.5),$

$(4.2, -1.5).$

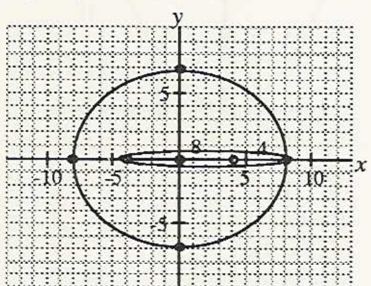
Endpoints of major axis:

$(2 \pm \sqrt{15}, -1\frac{1}{2})$; endpoints of minor axis: $(2, -1\frac{1}{2} \pm \sqrt{10})$



13. If the center of the ellipse is at (0, 0), the foci can be marked at (±4, 0). As shown in the diagram, the x-intercepts are 4" beyond the foci, at (±8, 0). Thus $a = 8$, $c = 4$.

$b = \sqrt{a^2 - c^2} = \sqrt{48} = 4\sqrt{3} \approx$ 6.9. Thus the equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $\frac{x^2}{64} + \frac{y^2}{48} = 1$.



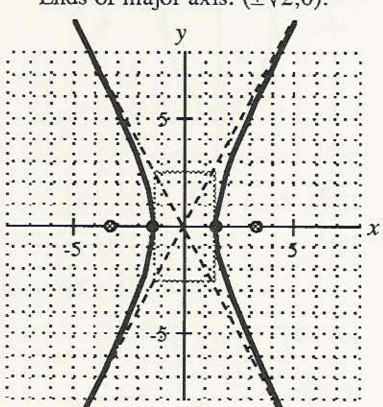
15. $4x^2 - y^2 = 8$

$\frac{x^2}{2} - \frac{y^2}{8} = 1$

$a = \sqrt{2} \approx 1.4$, $b = \sqrt{8} = 2\sqrt{2} \approx 2.8$, $c = \sqrt{10} \approx 3.2$

Center at (0, 0), foci at $(\pm\sqrt{10}, 0)$.

Ends of major axis: $(\pm\sqrt{2}, 0)$.



17. $y^2 + 2y - 4x^2 + 16x = 19$

$y^2 + 2y + 1 - 4(x^2 - 4x + 4)$

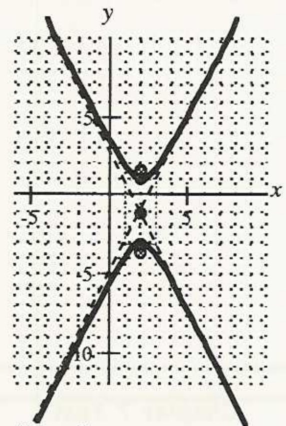
$= 19 + 1 - 4(4)$

$(y + 1)^2 - 4(x - 2)^2 = 4$

$\frac{(y + 1)^2}{4} - (x - 2)^2 = 1$

$a = 2, b = 1, c = \sqrt{5} \approx 2.2$

Center at (2, −1), foci at $(2, -1 \pm \sqrt{5})$



19. $9y^2 - 3x^2 - 18 = 0$

$\frac{y^2}{2} - \frac{x^2}{6} = 1$

Hyperbola, center at (0, 0).

21. $2x^2 + 2x - 2y^2 + 8y = 5$

$2(x^2 + x) - 2(y^2 - 4y) = 5$

$2(x^2 + x + \frac{1}{4}) - 2(y^2 - 4y + 4)$

$= 5 + \frac{1}{2} - 8$

$2(x + \frac{1}{2})^2 - 2(y - 2)^2 = -\frac{5}{2}$

$(y - 2)^2 - (x + \frac{1}{2})^2 = \frac{5}{4}$

$\frac{(y - 2)^2}{\frac{5}{4}} - \frac{(x + \frac{1}{2})^2}{\frac{5}{4}} = 1$

Hyperbola, center at $(-\frac{1}{2}, 2)$.

23. $x^2 + 8x + y^2 - 4y = 20$

$x^2 + 8x + 16 + y^2 - 4y + 4$

$= 20 + 16 + 4$

$(x + 4)^2 + (y - 2)^2 = 40$

Circle, center at (−4, 2).

25. $y = x^2 - 2x + 4$

$y = 2x + 1$

$2x + 1 = x^2 - 2x + 4$

$x^2 - 4x + 3 = 0$

$(x - 3)(x - 1) = 0$

$x = 1$................$y = 2(1) + 1 = 3$

$x = 3$................$y = 2(3) + 1 = 7$

Solution: (1, 3) and (3, 7)

27. $x^2 + 3y^2 - 8y - 2 = 0$

$y = x^2 - 2$

$x^2 = y + 2$

$(y + 2) + 3y^2 - 8y - 2 = 0$

$3y^2 - 7y = 0$

$y(3y - 7) = 0$

$y = 0$................$x^2 = 0 + 2$

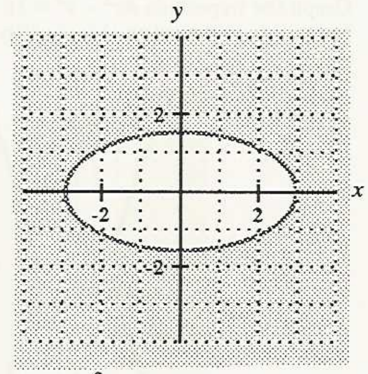
$x = \pm\sqrt{2}$

$y = \frac{7}{3}$................$x^2 = \frac{7}{3} + 2 = \frac{13}{3} = \frac{39}{9}$

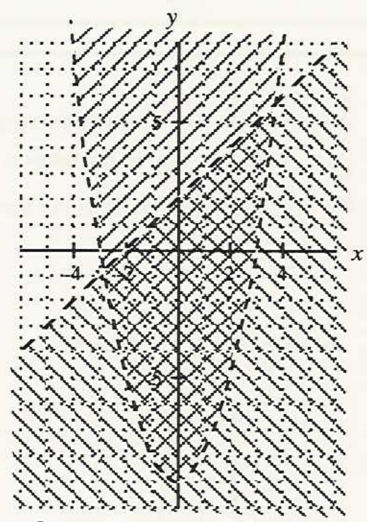
$x = \pm \frac{\sqrt{39}}{3}$

Solution: $(\sqrt{2}, 0)$, $(-\sqrt{2}, 0)$, $(\frac{\sqrt{39}}{3}, \frac{7}{3})$, $(-\frac{\sqrt{39}}{3}, \frac{7}{3})$

29. $x^2 + 3y^2 > 9$



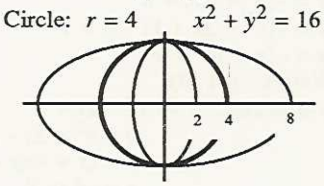
31. $y > x^2 - 9$
$y < x + 2$



33. Outer ellipse: $a = 8$, $b = 4$: $\frac{x^2}{64} + \frac{y^2}{16} = 1$

Inner ellipse: $a = 2$, $b = 4$: $\frac{x^2}{4} + \frac{y^2}{16} = 1$

Circle: $r = 4$ $\quad x^2 + y^2 = 16$



Love The Taste.
Taste The Love.

At Culver's® we can't think of anything better than serving up our creamy frozen custard and delicious classics cooked fresh the minute you order them. Which is why when we bring them to your table, they're always accompanied by a warm smile and a friendly offer to see if there's anything else we can get for you. So come on into your neighborhood Culver's and see for yourself. You might just be in love by the time you leave.

www.culvers.com

Chapter 12

Exercise 12–1

1. $a_n = \frac{5}{2}n - 3$
$\frac{5}{2}(1) - 3, \frac{5}{2}(2) - 3, \frac{5}{2}(3) - 3,$
$\frac{5}{2}(4) - 3, \ldots$
$-\frac{1}{2}, 2, \frac{9}{2}, 7, \ldots$

5. $a_n = 3$
$3, 3, 3, 3, \ldots$

9. $a_n = \dfrac{\sqrt{n}}{n + 1}$
$\frac{\sqrt{1}}{1 + 1}, \frac{\sqrt{2}}{2 + 1}, \frac{\sqrt{3}}{3 + 1}, \frac{\sqrt{4}}{4 + 1} \cdots$
$\frac{1}{2}, \frac{\sqrt{2}}{3}, \frac{\sqrt{3}}{4}, \frac{2}{5}, \ldots$

13. $2, 5, 8, 11, \ldots$
$2 + 0, 2 + 3, 2 + 6, 2 + 9, \ldots$
$2 + (n - 1)(3)$
$3n - 1$

17. $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4} \cdots$
$(-1)^n\left(\frac{1}{n}\right)$

21. $\frac{1}{4}, -\frac{4}{9}, \frac{9}{16}, -\frac{16}{25}, \ldots$
$\frac{1}{2^2}, -\frac{2^2}{3^2}, \frac{3^2}{4^2}, -\frac{4^2}{5^2}, \ldots$
$(-1)^{n+1}\left(\frac{n^2}{(n+1)^2}\right)$

25. The sequence 300, 400, 530, 710, … is definitely not an arithmetic sequence since the difference between terms is increasing. We therefore guess that it is a geometric sequence. The ratios of successive terms is $\frac{400}{300} = 1\frac{1}{3} \approx 1.33$, $\frac{530}{400} = 1\frac{13}{40} \approx 1.325$, $\frac{710}{530} = 1\frac{18}{53} \approx 1.34$. It seems reasonable to assume a constant ratio of $1\frac{1}{3}$, and therefore to estimate the next measurement as $710(\frac{4}{3}) = 947$, or about 950.

29. $1, \quad 0, \quad -1, \quad 0, \quad \ldots$
$0 - 1 = -1, -1 - 0 = -1, 0 - (-1) = 1,$ so there is no constant difference. There is no constant ratio possible since division by 0 is not defined. Thus, this is neither an arithmetic nor geometric sequence.

33. There is no constant difference or ratio, so this sequence is neither arithmetic nor geometric.

37. Since there is no constant difference or ratio this sequence is neither arithmetic nor geometric.

41. $-20, -16, -12, \ldots$ arithmetic; $d = 4$

45. $\frac{1}{6}, \frac{2}{11}, \frac{3}{16}, \frac{4}{21}, \ldots$ neither

49. $1, -1, 1, -1 \ldots$ geometric with $r = -1$

> **General term of an arithmetic sequence**
> If a is an arithmetic sequence with first term a_1 and common difference d, then the general term, a_n, is
> $$a_n = a_1 + (n-1)d.$$

53. $a_1 = 7$ and $d = 4$. $135 = 7 + (n - 1)(4)$
$n = 33$, so 135 is a_{33} and there are 33 terms in the sequence.

57. $a_{40} = a_1 + 39d$
$300 = 6 + 39d$
$294 = 39d$
$d = \frac{98}{13}$. $a_{30} = 6 + 26(\frac{98}{13}) = 202.$

61. $a_{15} = a_1 + 14d$
$49 = a_1 + 14d$
$a_{28} = a_1 + 27d$
$88 = a_1 + 27d.$
Thus, $a_1 = 49 - 14d$
and $a_1 = 88 - 27d$
so $49 - 14d = 88 - 27d$
$13d = 39$
$d = 3$. $a_1 = 88 - 27d$
so $a_1 = 88 - 81 = 7$. Thus, $a_4 = a_1 + 3d = 7 + 3(3) = 16.$

> **General term of a geometric sequence**
> If a is a geometric sequence with first term a_1 and common ratio r, then the general term, a_n, is
> $$a_n = a_1r^{n-1}.$$

65. $a_1 = -12, r = -\frac{1}{2}, n = 5$
$a_5 = -12(-\frac{1}{2})^4 = -12(\frac{1}{16}) = -\frac{3}{4}$

69. $a_5 = a_1r^4, \frac{5}{81} = 5r^4, \frac{1}{81} = r^4,$
$r =$ \r(4,\f(1,81)) = \f(1,3), so $a_4 = a_1r^3$
$= 5(\frac{1}{3})^3 = \frac{5}{27}$

73. $a_3 = a_2r, a_4 = a_2r^2$, so $5 = \frac{5}{4}r^2, 4 = r^2, r = 2$. $a_2 = a_1r$ so $\frac{5}{4} = a_1(2), a_1 = \frac{5}{8}$

77. $15000(1 - r)^6 = 3000$
$(1 - r)^6 = \frac{3000}{15000}$
$(1 - r)^6 = \frac{1}{5} = 0.2$
$1 - r = \sqrt[6]{0.2}$
$r = 1 - \sqrt[6]{0.2} \approx 0.23528$
$r \approx 23.5\%$

In problems 78, 79, 80 $a_n = 2n + 5$, and $b_n = 1 - n$.

81. Yes. We know that $a_n = a_1 + (n - 1)d_a$ for some constant d_a, and that
$b_n = b_1 + (n - 1)d_b$ for some constant d_b. Thus
$c_n = a_n + b_n$
$= a_1 + (n - 1)d_a + b_1 + (n - 1)d_b$
$= a_1 + b_1 + (n - 1)(d_a + d_b).$
By definition $a_1 + b_1 = c_1$, and let
$d_c = d_a + d_b$, a constant , so that
$c_n = c_1 + (n - 1)dc$, which is an arithmetic sequence.

In problems 84, 85, 86 $a_n = 3^n$ and $b_n = 3(2^n)$.

85. (a) sequence a: $3, \quad 9, \quad 27, \quad 81$
sequence d: $\frac{3}{2}, \quad \frac{9}{2}, \quad \frac{27}{2}, \quad \frac{81}{2}$
(b) Yes; $d_1 = \frac{3}{2}, r = 3.$
(c) $d_n = \frac{1}{2}a_n = \frac{1}{2}(3^n)$, or $d_n = \dfrac{3^n}{2}$.
$d_5 = \frac{3^5}{2} = \frac{243}{2}.$

89. Yes. Let
$c_n = (a_n)(b_n)$
$= [a_1(r_a)^{n-1}][b_1(r_b)^{n-1}]$
$= (a_1b_1)(r_ar_b)^{n-1}$
Since $a_1, b_1, r_a, r_b \neq 0$, then $a_1b_1 \neq 0$ and $r_ar_b \neq 0$, so c_n is a geometric sequence with first term a_1b_1 and ratio r_ar_b.

93. (a) $9, 6, 4, 2\frac{2}{3}, \ldots$ (Geometric sequence with $a_1 = 9, r = \frac{2}{3}$)
substitute into $b_n = an^2 + b_n + c$:
$n = 1$: $\quad 9 = a + b + c$
$n = 2$: $\quad 6 = 4a + 2b + c$
$n = 3$: $\quad 4 = 9a + 3b + c.$
$a = \frac{1}{2}, b = -\frac{9}{2}, c = 13$, so $b_n = \frac{1}{2}n^2 - \frac{9}{2}n + 13$, and
$b_4 = \frac{1}{2}(16) - \frac{9}{2}(4) + 13 = 3$, not $2\frac{2}{3} = a_4$.
(b) $3, 2, \frac{2}{3}, \frac{4}{9}, \ldots$ (Geometric sequence with $a_1 = 3, r = \frac{2}{3}$)
substitute into $b_n = an^2 + b_n + c$:
$n = 1$: $\quad 3 = a + b + c$
$n = 2$: $\quad 2 = 4a + 2b + c$
$n = 3$: $\quad \frac{2}{3} = 9a + 3b + c.$

$a = -\frac{1}{6}, b = -\frac{1}{2}, c = \frac{11}{3}$, so $b_n = -\frac{1}{6}n^2 - \frac{1}{2}n + \frac{11}{3}$, and
$b_4 = -\frac{1}{6}(16) - \frac{1}{2}(4) + \frac{11}{3} = -1$, not $\frac{4}{9}$, which is a_4.

(c) 3, 1, 4, 1, …
($a_n = n$'th term in the decimal expansion of π.)

substitute into $b_n = an^2 + bn + c$:

$n = 1$: $3 = a + b + c$
$n = 2$: $1 = 4a + 2b + c$
$n = 3$: $4 = 9a + 3b + c$.

$a = \frac{5}{2}, b = -\frac{19}{2}, c = 10$, so $b_n = \frac{5}{2}n^2 - \frac{19}{2}n + 10$, and
$b_4 = \frac{5}{2}(16) - \frac{19}{2}(4) + 10 = 12$, not 1, which is a_4.

Exercise 12–2

1. $(4(1) + 1) + (4(2) + 1) + (4(3) + 1) + (4(4) + 1)$
$5 + 9 + 13 + 17$

5. $\frac{3}{3+1} + \frac{4}{4+1} = \frac{3}{4} + \frac{4}{5}$

9. $\sum_{k=1}^{1} k^2 + \sum_{k=1}^{2} k^2 + \sum_{k=1}^{3} k^2 + \sum_{k=1}^{4} k^2$
$1 + (1 + 4) + (1 + 4 + 9) + (1 + 4 + 9 + 16)$

Sum of the first n terms of an arithmetic sequence
The sum S_n of the first n terms of an arithmetic sequence with first term a_1, n'th term a_n is $S_n = \frac{n}{2}(a_1 + a_n)$. An equivalent formula is $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$.

13. $-10, -14, -18, \ldots, -66$
$56, 52, 48, \ldots 0$ Add 66 to each term.
$14, 13, 12, \ldots 0$ Divide each term by 4.
There are 15 terms.
$S_{15} = \frac{15}{2}(-10 + (-66)) = 15(-38) = -570$

17. $a_1 = 3, d = -5$; find S_{12}
$a_{12} = 3 + 11(-5) = -52$;
$S_{12} = \frac{12}{2}(3 + (-52)) = 6(-49) = -294$

21. $4, -2, -8, \ldots$; find S_{14}
$a_1 = 4, d = -6$, so $a_{14} = 4 + 13(-6) = -74$;
$S_{14} = \frac{14}{2}(4 + (-74)) = 7(-70) = -490$

25. $a_5 = 50, a_8 = 68$; find S_6
$a_6 = a_5 + d; a_7 = a_5 + 2d; a_8 = a_5 + 3d$
$68 = 50 + 3d$
$d = 6$
$a_5 = a_1 + (5 - 1)d$
$50 = a_1 + 4 \cdot 6$
$a_1 = 26$
$S_n = \frac{n}{2}[2a_1 + (n - 1)d]$
$S_6 = \frac{6}{2}[2a_1 + (6 - 1)d]$
$= 3(52 + 5 \cdot 6) = 246$

Sum of the first n terms of a geometric sequence
The n'th partial sum S_n of the first n terms of a geometric sequence with first term a_1 and ratio r, $r < 1$ is $S_n = \dfrac{a_1(1 - r^n)}{1 - r}$.

29. $a_1 = -\frac{1}{4}, r = 1\frac{1}{4}$; find S_4
$$S_4 = \frac{-\frac{1}{4}(1 - (\frac{5}{4})^4)}{1 - \frac{5}{4}} = 1 - (\frac{5}{4})^4 = 1 - \frac{625}{256} = -\frac{369}{256}$$

33. $\frac{8}{27}, \frac{4}{9}, \frac{2}{3}, \ldots$; find S_9
$$S_9 = \frac{\frac{8}{27}(1 - (\frac{3}{2})^9)}{1 - (\frac{3}{2})} = \frac{\frac{8}{27} - \frac{8}{27}(\frac{39}{29})}{-\frac{1}{2}} =$$
$$-2(\frac{8}{27} - \frac{23}{33}(\frac{39}{29})) = -2(\frac{8}{27} - \frac{36}{26}) = -2(\frac{8}{27} - \frac{729}{64})$$
$$= -2(-\frac{19171}{1728}) = \frac{19171}{864} = 22\frac{163}{864}$$

37. $\sum_{k=1}^{6} \frac{1}{9}(3)^k = \frac{1}{3} + 1 + 3 + 9 + 27 + 81$
$a_1 = \frac{1}{3}, r = 3, n = 6$:
$$S_n = \frac{\frac{1}{3}(1 - 36)}{1 - 3} = \frac{\frac{1}{3}(-728)}{-2} = \frac{1}{3}(364) = 121\frac{1}{3}$$

41. $\sum_{k=1}^{12} \frac{16}{3}(\frac{1}{2})^k = \frac{8}{3} + \frac{4}{3} + \ldots + \frac{1}{3(2^8)}$; $a_1 = \frac{8}{3}$ $r = \frac{1}{2}, n = 12$:
$$S_n = \frac{\frac{8}{3}(1 - (\frac{1}{2})12)}{1 - (\frac{1}{2})} = 2(\frac{8}{3})(1 - \frac{1}{4096}) = \frac{16}{3}(\frac{4095}{4096}) = \frac{1365}{256} = 5\frac{85}{256}$$

If $|r| < 1$ the sum of the terms of an infinite geometric series, denoted by S, is $S = \dfrac{a_1}{1 - r}$. If $|r| \geq 1$ the sum is not defined.

45. $\sum_{i=1}^{\infty} \frac{2}{3}(\frac{1}{3})^i$; $a_1 = \frac{2}{9}, r = \frac{1}{3}$. Since $|r| < 1$ $S = \dfrac{\frac{2}{9}}{1 - \frac{1}{3}} = \frac{2}{9} \cdot \frac{3}{2} = \frac{1}{3}$

49. $\sum_{i=1}^{\infty} (\frac{4}{3})^i$; $a_1 = r = \frac{4}{3}$. $|r| \geq 1$ so the sum is not defined.

53. $1 - \frac{2}{3} + \frac{4}{9} - \ldots$;
$a_1 = 1, r = -\frac{2}{3}$. Since $|r| < 1$ $S = \dfrac{1}{1 - (-\frac{2}{3})} = 1 \cdot \frac{3}{5} = \frac{3}{5}$

57. $x = 0.2828\overline{28}$
$100x = 28.2828\overline{28}$
$x = 0.2828\overline{28}$
$99x = 28$
$x = \frac{28}{99}$.

61. $x = 0.51555155\overline{5155}$
$10000x = 5155.51555155\overline{5155}$
$x = 0.51555155\overline{5155}$
$9999x = 5155$
$x = \frac{5155}{9999}$.

65. $x = 0.401414\overline{14}$
$10000x = 4014.14\overline{14}$
$100x = 40.1414\overline{14}$
$9900x = 3974$
$x = \frac{3974}{9900} = \frac{1987}{4950}$.

69. We want S_8 for an arithmetic progression with $a_1 = 16$, $a_8 = 240$. $S_8 = \frac{8}{2}(16 + 240) = 4(256) = 1024$ feet.

73. Let a_1 be the original amount of the culture.
After 1 hour the culture has grown to $1.12a_1$.
After 2 hours the culture has grown to $1.12(1.12a_1) = 1.12^2a_1$.
After n hours the culture has grown to 1.12^na_1.
We want the value of n such that $1.12^na_1 = 2a_1$. Thus
$1.12^na_1 = 2a_1$

$1.12^n = 2$

We can find n by trial and error: $1.12^6 \approx 1.97$, and $1.12^7 \approx 2.21$, so that $n \approx 6$. We could also employ logarithms to solve the problem:

$$2 = 1.12^n$$
$$\log 2 = \log 1.12^n$$
$$\log 2 = n \log 1.12$$
$$\frac{\log 2}{\log 1.12} = n$$
$$n = \frac{\log 2}{\log 1.12}$$
$$n \approx 6.1.$$

Thus the population of bacteria will double in a little over 6 hours.

77. We have an arithmetic series with $a_1 = 3$, $d = 2$. We want the largest n such that $400 \geq S_n$

$$400 \geq \frac{n}{2}[2a_1 + (n-1)d]$$
$$800 \geq n[6 + 2(n-1)]$$
$$0 \geq 2n^2 + 4n - 800$$
$$0 \geq n^2 + 2n - 400.$$

The positive zero of $n^2 + 2n - 400$ is $n \approx 19.02$; this is one critical point, also.

$S_{19} = \frac{19}{2}[6 + 18(2)] = 399$, and $S_{20} = \frac{20}{2}[6 + 19(2)] = 420$, so the clerk should use 19 rows. $a_{19} = 3 + 18(2) = 39$, so the clerk should put 39 boxes in the first row, which will use 19 rows and 399 boxes.

81. This is an arithmetic series with $a_1 = 1$, $d = \frac{1}{4}$, and we want

$S_{250} = \frac{250}{2}[2(1) + 249(\frac{1}{4})] = 125(\frac{257}{4}) = \$8031.25.$

85. Add up the integers from 1 to 100. This is an arithmetic series.

$$S_n = \frac{n}{2}(a_1 + a_n),\ n = 100,\ a_1 = 1,\ a_{100} = 100.$$
$$S_{100} = \frac{100}{2}(1 + 100) = 50(101) = 5{,}050.$$

89. We sum the series $\sum_{i=1}^{10} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{10}$. This series is neither geometric nor arithmetic, so we can only do it by "brute force" – that is, just do the indicated calculations. This turns out to be about 2.9. Thus one would expect about 3 record snowfalls in the first 10 years of life.

Exercise 12–3

1. $\frac{7!}{5!} = \frac{7 \cdot 6 \cdot 5!}{5!} = 7 \cdot 6 = 42$

5. $\binom{6}{6} = \frac{6!}{6!0!} = 1$

> The Binomial Expansion Theorem: $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i,\ n \in N.$

9. $(ab - 3)^4 =$

$\binom{4}{0}(ab)^4(-3)^0 + \binom{4}{1}(ab)^3(-3)^1 + \binom{4}{2}(ab)^2(-3)^2 + \binom{4}{3}(ab)(-3)^3 + \binom{4}{0}(ab)^0(-3)^4 =$

$a^4b^4 + 4a^3b^3(-3) + 6a^2b^2(9) + 4ab(-27) + 81 =$

$a^4b^4 - 12a^3b^3 + 54a^2b^2 - 108ab + 81$

13. $(a^3b^2 - 2^c)^7 = \binom{7}{0}(a^3b^2)^7(-2c)^0 + \binom{7}{1}(a^3b^2)6(-2c)^1 + \binom{7}{2}(a^3b^2)^5(-2c)^2 + \binom{7}{3}(a^3b^2)^4(-2c)^3 + \binom{7}{4}(a^3b^2)^3(-2c)^4 + \binom{7}{5}(a^3b^2)^2(-2c)^5$
$+ \binom{7}{6}(a^3b^2)^1(-2c)^6 + \binom{7}{7}(a^3b^2)^0(-2c)^7$

$= a^{21}b^{14} + 7(a^{18}b^{12})(-2c) + 21(a^{15}b^{10})(4c^2) + 35(a^{12}b^8)(-8c^3) + 35(a^9b^6)(16c^4) + 21(a^6b^4)(-32c^5) + 7(a^3b^2)(64c^6) + (-128c^7)$

$= a^{21}b^{14} - 14a^{18}b^{12}c + 84a^{15}b^{10}c^2 - 280a^{12}b^8c^3 + 560a^9b^6c^4 - 672a^6b^4c^5 + 448a^3b^2c^6 - 128c^7$

17. Find the 5'th term of $(a^3 + 2b^5)^{15}$. The 5'th term is when $i = 4$:

$\binom{15}{4}(a^3)^{15-4}(2b^5)^4 = 1365a^{33}(2^4b^{20}) = 21{,}840a^{33}b^{20}$

> Sum of Constants Property: $\sum_{i=1}^{n} k = nk$
>
> Constant Factor Property: $\sum_{i=1}^{n}[k \cdot f(i)] = k \cdot \sum_{i=1}^{n} f(i)$
>
> Sum of Terms Property: $\sum_{i=1}^{n}[f(i) + g(i)] = \sum_{i=1}^{n} f(i) + \sum_{i=1}^{n} g(i)$
>
> Sum of Integer Series: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
>
> Sum of Squares of Integers Series: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
>
> Sum of Cubes of Integers Series: $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

21. $\sum_{i=1}^{56} 8 = 8(56) = 448$ Sum of Constants

25. $\sum_{i=1}^{9}(3 - 4i + i^2)$

$= \sum_{i=1}^{9} 3 - 4\sum_{i=1}^{9} i + \sum_{i=1}^{9} i^2 = 9(3) - 4\left(\frac{9(10)}{2}\right) + \frac{9(10)(19)}{6} = 132$

29. $\sum_{i=1}^{7}(2i^3 - 3) = \sum_{i=1}^{7}(2i^3 - 3)$

$= 2\sum_{i=1}^{7} i^3 - \sum_{i=1}^{7} 3$

$= 2\left(\frac{7(8)}{2}\right)^2 - 7(3) = 1{,}547$

33. $\sum_{i=1}^{4}[i^2 - (\frac{1}{4})^i] = \sum_{i=1}^{4} i^2 - \sum_{i=1}^{4}(\frac{1}{4})^i$

The second expression is a geometric series, $a_1 = r = \frac{1}{4}$;

$S_4 = a_4\left(\frac{1 - r^4}{1 - r}\right)$

$\frac{4(5)(9)}{6} - \frac{1}{4}\left(\frac{1 - (\frac{1}{4})^4}{1 - \frac{1}{4}}\right)$

$= 30 - \frac{1}{4}\left(\frac{1 - \frac{1}{256}}{\frac{3}{4}}\right) = 30 - \frac{1}{4} \cdot \frac{4}{3}\left(1 - \frac{1}{256}\right)$

$= 30 - \frac{1}{3}\left(\frac{255}{256}\right) = 30 - \frac{85}{256} = 29\frac{171}{256}.$

37. $\sum_{i=1}^{k}[6i^2 - 4i + 2] = 6\sum_{i=1}^{k} i^2 - 4\sum_{i=1}^{k} i + \sum_{i=1}^{k} 2$

$= 6 \cdot \frac{k(k+1)(2k+1)}{6} - 4 \cdot \frac{k(k+1)}{2} + 2 \cdot k$

$= k(k+1)(2k+1) - 2k(k+1) + 2k$

$= 2k^3 + 3k^2 + k - 2k^2 - 2k + 2k = 2k^3 + k^2 + k$

41.
```
                1   1
              1   2   1
            1   3   3   1
          1   4   6   4   1
        1   5  10  10   5   1
      1   6  15  20  15   6   1
    1   7  21  35  35  21   7   1
  1   8  28  56  70  56  28   8   1
```

45. $\sum_{i=1}^{n} i = 1 + 2 + 3 \ldots + n$ is an arithmetic sequence with $a_1 = 1$, $a_n = n, n = n$.

$S_n = \frac{n}{2}(a_1 + a_n)$

$S_n = \frac{n}{2}(1+n) = \frac{n(n+1)}{2}.$

49. Using $(x+y)^n = \sum_{i=0}^{n}\binom{n}{i}x^{n-i}y^i$ with $x = y = 1$ we obtain:

$2^n = (1+1)^n = \sum_{i=0}^{n}\binom{n}{i}1^{n-i}1^i = \sum_{i=0}^{n}\binom{n}{i}.$

Exercise 12–4

1. $2 + 4 + 6 + \ldots + 2n = n(n+1)$

Show true for $n = 1$: $2(1) = 1(1+1)$; $2 = 2$ ✔

Find goal statement:

$2 + 4 + 6 + \ldots + 2(k+1) = (k+1)[(k+1)+1]$ Replace n by $k + 1$.

$2 + 4 + 6 + \ldots + 2(k+1) = \mathbf{(k+1)(k+2)}$ Goal statement.

Assume true for $n = k$:

$2 + 4 + 6 + \ldots + 2k = k(k+1)$ Replace n by k.

$2 + 4 + 6 + \ldots + 2k + \mathbf{2(k+1)} = k(k+1) + \mathbf{2(k+1)}$

Add next term to both members.

$= k(k+1) + 2(k+1)$ Common factor.

$= (k+1)(k+2)$ ✔

This is the goal statement, so the proposition is true.

5. $1 + 5 + 9 + \ldots + (4n - 3) = 2n^2 - n$

Show true for $n = 1$: $(4(1) - 3) = 2(1)^2 - 1$; $1 = 1$ ✔

Find the goal statment:

$1 + 5 + 9 + \ldots + (4(k+1) - 3) = 2(k+1)^2 - (k+1)$ Replace n by $k + 1$.

$= \mathbf{2k^2 + 3k + 1}$ Goal statement.

Assume true for $n = k$:

$1 + 5 + 9 + \ldots + (4k - 3) = 2k^2 - k$ Replace n by k.

$1 + 5 + 9 + \ldots + (4k - 3) + \mathbf{(4(k+1) - 3)}$

$= 2k^2 - k + \mathbf{(4(k+1) - 3)}$

Add next term to both members.

$= 2k^2 + 3k + 1$ ✔

This is the goal statement, so the proposition is true.

9. $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$

Show true for $n = 1$: $\frac{1}{2^1} = \frac{2^1 - 1}{2^1}$ $\frac{1}{2} = \frac{1}{2}$ ✔

Find goal statement:

$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^{k+1}} = \frac{\mathbf{2^{k+1} - 1}}{\mathbf{2^{k+1}}}$ Replace n by $k + 1$.

Assume true for $n = k$:

$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^k} = \frac{2^k - 1}{2^k}$ Replace n by k.

$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^k} + \frac{\mathbf{1}}{\mathbf{2^{k+1}}} = \frac{2^k - 1}{2^k} + \frac{\mathbf{1}}{\mathbf{2^{k+1}}}$ Add next term to each member.

$= \frac{2(2^k - 1)}{2(2^k)} + \frac{1}{2^{k+1}} = \frac{2^{k+1} - 1}{2^{k+1}}$ ✔

This is the goal statement, so the proposition is true.

13. $\frac{1}{1\cdot3} + \frac{1}{3\cdot5} + \frac{1}{5\cdot7} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

Show true for $n = 1$: $\frac{1}{(2(1)-1)(2(1)+1)} = \frac{1}{2(1)+1}$; $\frac{1}{3} = \frac{1}{3}$ ✔

Find the goal statement:

$\frac{1}{1\cdot3} + \frac{1}{3\cdot5} + \frac{1}{5\cdot7} + \ldots + \frac{1}{(2(k+1)-1)(2(k+1)+1)} = \frac{k+1}{2(k+1)+1} = \frac{\mathbf{k+1}}{\mathbf{2k+3}}$

Replace n by $k + 1$.

Assume true for $n = k$:

$\frac{1}{1\cdot3} + \frac{1}{3\cdot5} + \frac{1}{5\cdot7} + \ldots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}$ Replace n by k.

$\frac{1}{1\cdot3} + \frac{1}{3\cdot5} + \frac{1}{5\cdot7} + \ldots + \frac{1}{(2k-1)(2k+1)} + \frac{\mathbf{1}}{\mathbf{(2(k+1)-1)(2(k+1)+1)}}$

Add the next term to both members.

$= \frac{k}{2k+1} + \frac{\mathbf{1}}{\mathbf{(2(k+1)-1)(2(k+1)+1)}} = \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$

$= \frac{k(2k+3)}{(2k+1)(2k+3)} + \frac{1}{(2k+1)(2k+3)} = \frac{2k^2 + 3k + 1}{(2k+1)(2k+3)}$

$= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} = \frac{k+1}{2k+3}$ ✔

17. $8 + 4 + 2 + \ldots + \frac{1}{2^{n-4}} = \frac{2^n - 1}{2^{n-4}}$

Show true for $n = 1$: $\frac{1}{2^{-3}} = \frac{2^1 - 1}{2^{-3}}$; $8 = 8$ ✔

Find goal statement:

$8 + 4 + 2 + \ldots + \frac{1}{2^{(k+1)-4}} = \frac{\mathbf{2^{k+1} - 1}}{\mathbf{2^{(k+1)-4}}}$ Replace n by $k + 1$.

Assume true for $n = k$:

$8 + 4 + 2 + \ldots + \frac{1}{2^{k-4}} = \frac{2^k - 1}{2^{k-4}}$ Replace n by k.

$8 + 4 + 2 + \ldots + \frac{1}{2^{k-4}} + \frac{\mathbf{1}}{\mathbf{2^{(k+1)-4}}} = \frac{2^k - 1}{2^{k-4}} + \frac{\mathbf{1}}{\mathbf{2^{(k+1)-4}}}$

Add next term to both members.

$= \frac{2^k - 1}{2^{k-4}} + \frac{1}{2^{k-3}} = \frac{2}{2}\cdot\frac{2^k - 1}{2^{k-4}} + \frac{1}{2^{k-3}}$

$= \frac{2^{k+1} - 2}{2^{k-3}} + \frac{1}{2^{k-3}} = \frac{2^{k+1} - 1}{2^{k-3}} = \frac{2^{k+1} - 1}{2^{(k-4)+1}}$ ✔ $2\cdot2^{k-4} = 2^{(k-4)+1}$

21. $\frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + \frac{1}{3\cdot4\cdot5} + \ldots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$

Show true for $n = 1$: $\frac{1}{1\cdot2\cdot3} = \frac{1(4)}{4(2)(3)}$; $\frac{1}{6} = \frac{1}{6}$ ✔

Find goal statement:

$\frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + \frac{1}{3\cdot4\cdot5} + \ldots + \frac{1}{(k+1)((k+1)+1)((k+1)+2)} = \frac{(k+1)((k+1)+3)}{4((k+1)+1)((k+1)+2)}$

$= \frac{\mathbf{(k+1)(k+4)}}{\mathbf{4(k+2)(k+3)}}$

Assume true for $n = k$:

$\frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + \frac{1}{3\cdot4\cdot5} + \ldots + \frac{1}{k(k+1)(k+2)} = \frac{k(k+3)}{4(k+1)(k+2)}$

$\frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + \frac{1}{3\cdot4\cdot5} + \ldots + \frac{1}{k(k+1)(k+2)} + \frac{\mathbf{1}}{\mathbf{(k+1)(k+2)(k+3)}}$

= \f(k(k+3),4(k+1)(k+2)) +

$\frac{\mathbf{1}}{\mathbf{(k+1)(k+2)(k+3)}} = \frac{k(k+3)^2}{4(k+1)(k+2)(k+3)} + \frac{4}{4(k+1)(k+2)(k+3)}$

$= \frac{k^3 + 6k^2 + 9k + 4}{4(k+1)(k+2)(k+3)} = \frac{(k+1)^2(k+4)}{4(k+1)(k+2)(k+3)} = \frac{(k+1)(k+4)}{4(k+2)(k+3)}$ ✔

Using the goal statement and the rational zero theorem as a guide factor the numerator.

This is the goal statement, so the proposition is true.

25. Goal statement:

$1 + 3 + 5 + \ldots + (2(k+1) - 1) = \frac{(k+1)^2 + (k+1)}{2} = \frac{k^2 + 3k + 2}{2}$

Assume true for $n = k$, then add the next term to both members.

$1 + 3 + 5 + \ldots + (2k - 1) = \frac{k^2 + k}{2}$

$1 + 3 + 5 + \ldots + (2k - 1) + \mathbf{(2(k+1) - 1)}$

$= \frac{k^2 + k}{2} + \mathbf{(2(k+1) - 1)}$

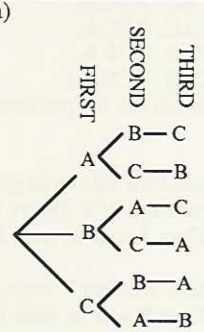
The left side is now the left side of the goal statment; we must show that the right side is the same as the right side of the goal statement.

$= \frac{k^2 + k}{2} + \frac{2(2k+1)}{2} = \frac{k^2 + 5k + 2}{2}$

This expression is clearly not the same as the goal expression.

Exercise 12–5

1. (a)



(b) ABC ACB BAC BCA CBA CAB

5. $4 \cdot 6 = 24$
9. $12 \cdot 15 = 180$
13. $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 5040$
17. $\frac{12!}{3!9!} = \frac{12 \cdot 11 \cdot 10 \cdot 9!}{3 \cdot 2 \cdot 9!} = 2 \cdot 11 \cdot 10 = 220$
21. ${}_6P_6 = 6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 720$
25. ${}_{20}P_1 = 20$
29. Order is important, so we are counting permutations: ${}_9P_3 = 504$.
33. $7! = 5040$
37. ${}_{15}P_9 = 1{,}816{,}214{,}400$.
41. There are a total of 12 symbols, of which there are 2 a's, 4 b's and 5 c's. Thus the number is $\frac{12!}{2!4!5!} = 83{,}160$.
45. ${}_8C_5 = \frac{8!}{5!(8-5)!} = \frac{8 \cdot 7 \cdot 6 \cdot 5!}{5! \cdot 3 \cdot 2} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2} = 56$
49. By definition, ${}_nC_n = \frac{n!}{n!(n-n)!} = \frac{n!}{n!0!}$ $= \frac{n!}{n!} = 1$
53. ${}_{12}C_8$ or ${}_{12}C_4 = 495$.
57. Every selection of three points determines a triangle; two examples are shown in the figure. The order of selection is not important. Thus, the number of triangles is ${}_7C_3 = 35$.

61. a) ${}_{17}C_9 = 24310$ b) We want to choose 2 players from 17, but the order is important, since one will be the captain and the other the co–captain. We want ${}_{17}P_2 = 272$. c) There are ${}_{17}C_9$ 9–player teams. For each team there are 9! batting orders. Thus the number of different batting orders is ${}_{17}C_9 \cdot 9! = 8{,}821{,}612{,}800$. d) ${}_{17}C_{12} = 6{,}188$.
65. a) After selecting a digit there are one fewer choices left for the next selection. Thus there are $5 \cdot 4 \cdot 3 = 60 = {}_5P_3$ 3–digit numbers under these circumstances. b) All of the 60 3–digit numbers end in 1,2,3,4 or 5. Those ending in 1,3,5 are odd. This is $\frac{3}{5}$'ths of the 60 or 36 numbers. Another way to see this is to count how many numbers end in 1, 3 or 5. If a number ends in 1, then the previous two digits were chosen from the set { 2, 3, 4, 5 }, which occurred in $4 \cdot 3 = 12$ ways. Thus there are 12 numbers ending in "1", 12 ending in "3" and 12 ending in "5", for 12 + 12 + 12 = 36 such numbers. c) $2 \cdot 3 \cdot 1 = 6$ d) ${}_3P_3 = 3! = 6$
69. a) ${}_{12}C_3 \cdot {}_{18}C_3 = 179{,}520$ b) ${}_{12}C_6 = 924$ c) There are 30 people from which a committee of 6 is to be selected: ${}_{30}C_6 = 593{,}775$
73. The order of selecting the horses in each race is important. For each of the ${}_9P_3$ trifecta selections in the first race one may make any of the ${}_8P_3$ trifecta selections in the second race. Thus, the number of double trifecta bets is ${}_9P_3 \cdot {}_8P_3 = 169{,}344$.
77. Let $n = 10$, $r = 3$.
${}_nP_r = n(n-1)(n-2) \cdots (n-[r-1])$
$n - (r-1) = 8$, so ${}_nP_r$ becomes ${}_{10}P_3 = 10 \cdot 9 \cdot 8$.
$\frac{n!}{(n-r)!}$ becomes $\frac{10!}{(10-3)!}$
$= \frac{10 \cdot 9 \cdot 8 \cdot 7!}{7!} = 10 \cdot 9 \cdot 8$.

Exercise 12–6

A coin is tossed 2 times.
$S =$ { HH, HT, TH, TT }; n(S) = 4

Find the probability of

1. exactly one head. $A =$ { HT, TH }.
$P(A) = \frac{n(A)}{n(S)} = \frac{2}{4} = \frac{1}{2}$

A coin is tossed 3 times.
$S =$ {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}; n(S) = 8

Find the probability of

5. all tails. $A =$ {TTT }; $\frac{1}{8}$

A coin is tossed 4 times.
$S =$ {HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT, THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT } ; n(S) = 16

Find the probability of (A shown in bold)

9. exactly two heads. $S =$ {HHHH, HHHT, HHTH, **HHTT**, HTHH, **HTHT**, **HTTH**, HTTT, THHH, **THHT**, **THTH**, THTT, **TTHH**, TTHT, TTTH, TTTT} $\frac{6}{16} = \frac{3}{8}$.

A card is drawn from a standard deck of playing cards.
$S =$

Color	Suit	Numbered Cards	Face Cards
Black	Clubs	Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10	Jack, Queen, King
Black	Spades	Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10	Jack, Queen, King
Red	Diamonds	Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10	Jack, Queen, King
Red	Hearts	Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10	Jack, Queen, King

What is the probability of

13. a ten. $\frac{4}{52} = \frac{1}{13}$
17. a card from 4 through 9, inclusive. $\frac{24}{52} = \frac{6}{13}$
21. a black 4 or 5 $\frac{4}{52} = \frac{1}{13}$

A card is drawn from a standard deck of playing cards. Find the probability that the card is

25. P(from 2 through 6, inclusive, or a spade)
= P(from 2 through 6 inclusive) + P(spade)
– P(spade from 2 through 6 inclusive)
$= \frac{20}{52} + \frac{13}{52} - \frac{5}{52} = \frac{28}{52} = \frac{7}{13}$
29. P(not from 4 through 10, inclusive)
$= 1 - P(4,5,6,7,8,9,10) = 1 - \frac{7}{13} = \frac{6}{13}$
33. P(not red) = P(black) = P(club or spade) $= \frac{13}{52} + \frac{13}{52} = \frac{1}{2}$

A roulette wheel contains the numbers from 1 through 36. Eighteen of these numbers are red, and the other eighteen are black. There are two more numbers, 0, and 00, which are green. The wheel is spun, and a ball allowed to fall on one of these 38 locations at random, as the wheel stops. n(S) = 38.

What is the probability that the ball will land on

37. a number which is not green?
$= 1 - P(\text{green}) = 1 - \frac{2}{38} = 1 - \frac{1}{19} = \frac{18}{19}$
41. an even, nonzero number?
$A = \{2,4,6,8,10,12,14,16,18,20,22,24,26,28,30,32,34,36\}$
$P(A) = \frac{18}{38} = \frac{9}{19}$

A bowl contains 24 balls. Six are red, 10 blue, and 8 white. If one ball is randomly selected what is the probability that the ball is

45. black? None are black, so n(A) = 0, $P(A) = \frac{0}{24} = 0$.

There are 18 alternators on hand. Ten are new, and 8 are remanufactured. If six of the alternators are chosen randomly for a shipment, what is the probability that the shipment

49. contains all new alternators?

a. How many ways can we choose six alternators from all 18? ${}_{18}C_6$; this is the number of all possible shipments (of six items) we could make from the 18 items.

b. How many ways can we choose six alternators from the ten new alternators? ${}_{10}C_6$; this is the number of the shipments of six items which contain only new alternators. If necessary to see this better imagine purposefully selecting shipments of six items which are all new. The choices in part (a) include all the choices in (b). The choices in (a) are the sample space.

$$P(\text{all new}) = \frac{\text{number of shipments in which all six items are new}}{\text{number of shipments of six items}}$$

$$= \frac{{}_{10}C_6}{{}_{18}C_6} = \frac{210}{18564} \approx 0.0113.$$

The sample space is all possible collections of 5 cards, drawn from 52 cards, without regard to order. $n(S) = {}_{52}C_5 = 2{,}598{,}960$.

53. All five cards are red.

A = Choose 5 red cards. There are 26 red cards, so 5 can be chosen in ${}_{26}C_5 = 65780$ ways.

$$P(A) = \frac{n(A)}{n(S)} = \frac{65780}{2598960} \approx 0.0253.$$

57. All of the cards are face cards. (See the previous problem.)

$$\frac{{}_{12}C_5}{2598960} = \frac{792}{2598960} = 0.0003$$

61. $P(\text{Four diamonds and one spade}) = \frac{{}_{13}C_4 \cdot {}_{13}C_1}{2598960} = \frac{9295}{2598960} \approx 0.0036.$

65. (a) $\frac{4}{50} = \frac{2}{25}$

(b) P(at least one is defective) = 1 – P(none are defective)

$$= 1 - \frac{\text{Number of ways to choose 6 good TV's}}{\text{Number of ways to choose 6 TV's}}$$

$$= 1 - \frac{{}_{46}C_6}{{}_{50}C_6} = 1 - \frac{9366819}{15890700} \approx 0.4105.$$

A doctor has four patients waiting, patient A, B, C and D. They do not have appointments, and arrived at the same time.

S = { ABCD, ABDC, ACBD, ACDB, ADBC, ADCB, BACD, BADC, BCAD, BCDA, BDAC, BDCA, CABD, CADB, CBAD, CBDA, CDAB, CDBA, DABC, DACB, DBAC, DBCA, DCAB, DCBA }

If the doctor chooses the order in which to see the the patients at random, what is the probability that

69. A and B are seen before C or D.

A and B must be taken first and second for this to happen.

If A is first, and B second, there are 2! = 2ways for C and D to be seen third and fourth. Thus A can be seen first and B second in 2 ways.

If B is first, and A second, there are also 2 ways for C and D to be seen third and fourth. Thus B can be seen first and A second in 2 ways.

Seeing A first, then B, and seeing B first, then A, are are mutually exclusive events, so they can happen in 2 + 2 = 4 ways. Thus the probability of this event is $\frac{4}{24} = \frac{1}{6}$.

73. $n = \frac{t}{\text{MTBF}} = \frac{3000}{2000} = 1.5$

$P(\geq 1{,}3000) = 1 - P(0) = 1 - P(0{,}3000)$

$= 1 - \frac{e^{-1.5} \times (1.5)^0}{0!} = 1 - e^{-1.5} \approx 1 - 0.223 \approx 0.777.$

77. 15% of 100,000 is 15,000. Thus, 15,000 workers use the drugs and 85,000 do not.

The test will detect 90% of these 15,000 users, or 0.9(15,000) = 13,500 drug users. It will also falsely detect 10% of the 85,000 who do not use drugs, or 8,500. Thus, it will report that 13,500 + 8,500 = 22,000 of the 100,000 workers use drugs.

If a worker tests positive but does not use drugs the worker is among that 22,000. Since 8,500 of these 22,000 do not use drugs, the probability is $\frac{8{,}500}{22{,}000} \approx 0.39$ that the worker does not use drugs. In other words there is about a 40% chance that the worker does not use drugs even when the test is positive.

Exercise 12-7

1. $a_n = \begin{cases} 3 \text{ if } n = 0 \\ a_{n-1} + 5 \text{ if } n > 0 \end{cases}$: 3, 8, 13, 18, 23, …, an arithmetic sequence with $a_1 = 3$, $d = 5$, so $a_n = 3 + 5(n-1)$; however since we want to index from $n = 0$ we increase the index n to $n + 1$, obtaining $a_n = 3 + 5[(n+1) - 1]$, or $a_n = 5n + 3$.

5. $a_n = \begin{cases} -2 \text{ if } n = 0, 3 \text{ if } n = 1 \\ 2a_{n-1} + 3a_{n-2} \text{ if } n > 1 \end{cases}$: –2, 3, 2(3) + 3(–2) = 0, 2(0) + 3(3) = 9, 2(9) + 3(0) =18, … or –2, 3, 0, 9, 18, … .

This is neither geometric nor arithmetic, so we try a recurrence relation.

$a_n = 2a_{n-1} + 3a_{n-2}$

$a_n - 2a_{n-1} - 3a_{n-2} = 0$

$x^n - 2x^{n-1} - 3^{n-2} = 0$; Replace n by 2.

$x^2 - 2x - 3 = 0$, so $x = 3$ or -1.

Then $a_n = A(3^n) + B(-1)^n$. We find A and B from a_0 and a_1.

n=0: $a_0 = -2 = A + B$

n=1: $a_1 = 3 = 3A - B$

Solving (for example by adding the two equations we find $1 = 4A$) we find $A = \frac{1}{4}$, $B = -\frac{9}{4}$, so a general term is

$a_n = \frac{1}{4}(3^n) - \frac{9}{4}(-1)^n$.

9. $a_n = \begin{cases} -2 \text{ if } n = 0, 4 \text{ if } n = 1 \\ 3a_{n-1} + 6a_{n-2} \text{ if } n > 1 \end{cases}$: –2, 4, 3(4) + 6(–2) = 0, 3(0) + 6(4) = 24, 3(24) + 6(0) = 72, … or –2, 4, 0, 24, 72, ….

$a_n = 3a_{n-1} + 6a_{n-2}$

$a_n - 3a_{n-1} - 6a_{n-2} = 0$

$x^n - 3x^{n-1} - 6x^{n-2} = 0$; let $n = 2$:

$x^2 - 3x - 6 = 0$; $x = \frac{3 \pm \sqrt{33}}{2}$, so $a_n = A(\frac{3+\sqrt{33}}{2})^n + B(\frac{3-\sqrt{33}}{2})^n$.

n=0: $a_0 = -2 = A + B$

n=1: $a_1 = 4 = A(\frac{3+\sqrt{33}}{2}) + B(\frac{3-\sqrt{33}}{2})$

$B = -2 - A$, so $4 = A(\frac{3+\sqrt{33}}{2}) + (-2 - A)(\frac{3-\sqrt{33}}{2})$,

$8 = A(3 + \sqrt{33}) + (-2 - A)(3 - \sqrt{33})$

$8 = 3A + A\sqrt{33} - 6 + 2\sqrt{33} - 3A + A\sqrt{33}$

$14 - 2\sqrt{33} = 2A\sqrt{33}$

$\frac{14 - 2\sqrt{33}}{2\sqrt{33}} = A$; $B = -2 - A = \frac{-4\sqrt{33}}{2\sqrt{33}} - \frac{14 - 2\sqrt{33}}{2\sqrt{33}} = \frac{-14 - 2\sqrt{33}}{2\sqrt{33}}$.

Thus $a_n = (\frac{14 - 2\sqrt{33}}{2\sqrt{33}})(\frac{3+\sqrt{33}}{2})^n + (\frac{-14 - 2\sqrt{33}}{2\sqrt{33}})(\frac{3-\sqrt{33}}{2})^n$.

13. $a_n = \begin{cases} 4 \text{ if } n = 0,\ 3 \text{ if } n = 1 \\ 2a_{n-1} - a_{n-2} \text{ if } n > 1 \end{cases}$

$a_n = 2a_{n-1} - a_{n-2}$

$a_n - 2a_{n-1} + a_{n-2} = 0$

$x^n - 2x^{n-1} + x^{n-2} = 0, n = 2$

$x^2 - 2x + 1 = 0$; $x = 1$ (multiplicity 2)

$a_n = A(1^n) + Bn(1^n) = A + nB$

n=0: $a_0 = 4 = A$

n=1: $a_1 = 3 = A + B$, so $B = -1$.

$a_n = 4 - n$ (an arithmetic series).

17. With a recursive definition, to compute a_n we need to first find some or all of the previous terms, $a_0, a_1, \ldots, a_{n-1}$.

21. For the given sequence, $a_2 = 2a_1 + 3a_0 = 2A + 6$. We thus know that $\frac{a_1}{a_0} = \frac{A}{2}$, and $\frac{a_2}{a_1} = \frac{2A+6}{A}$. We want these ratios to be equal, so we solve $\frac{A}{2} = \frac{2A+6}{A}$. $A^2 = 4A + 12$, $A^2 - 4A - 12 = 0$, $A = -2$ or 6. Both of these values do produce geometric sequences.

Chapter 12 Review

1. $a_n = 6n - 2$: $\quad 6(1) - 2, 6(2) - 2, 6(3) - 2, 6(4) - 2$

$\quad$ 4, 10, 16, 22

3. $a_n = (n-1)^2$ $\quad (1-1)^2, (2-1)^2, (3-1)^2, (4-1)^2$

$\quad$ 0, 1, 4, 9

5. $-200, -160, -120, \ldots$

$-200, -200 + 40, -200 + 80, \ldots$

$a_n = -200 + (n-1)(40)$

7.

300		350		425		525		n
	50		75		100		125	

Based upon the increasing differences, we might expect $525 + 125 = 650$ cars in the next 15 minutes.

9. The difference between terms is not constant, and there is not common ratio. Thus this sequence is neither geometric nor arithmetic.

11. $a_1 = 150, d = -2$.

$a_n = 150 + (n-1)(-2)$

$= 118$

$118 = 150 - 2n + 2$

$17 = n$

There are 17 terms in the sequence.

13. $a_1 = 1024, r = -\frac{1}{2}, n = 5$

$a_5 = 1024(-\frac{1}{2})^4$

$= 1024 \cdot \frac{1}{16} = 64$

15. $a_1 = 1, r = 0.1, n = 3$ $\quad a_3 = 1 \cdot 0.1^2 = 0.01$

17. $a_5 = a_1r^4$, and $a_3 = a_1r^2$, so $\frac{a_5}{a_3} = \frac{a_1r^4}{a_1r^2} = r^2$. Thus $\frac{5}{\frac{5}{4}} = 4 = r^2$, so $r = \pm 2$, so $r = 2$. $a_6 = a_5r = 5 \cdot 2$. Thus $a_6 = 10$.

19.

1'st	2'nd	3'rd	4'th	n'th
$\frac{2}{3}\cdot 12$	$\frac{2}{3}\cdot 8$	$\frac{2}{3}\cdot\frac{16}{3}$	$\frac{2}{3}\cdot\frac{32}{9}$	**$12 \cdot (\frac{2}{3})^n$ (c)**
8	**$\frac{16}{3}$ (a)**	$\frac{32}{9}$	**$\frac{64}{27}$ (b)**	

21. $\frac{2(1)+1}{1+1} + \frac{2(2)+1}{2+1} + \frac{2(3)+1}{3+1} + \frac{2(4)+1}{4+1} + \frac{2(5)+1}{5+1} = \frac{3}{2} + \frac{5}{3} + \frac{7}{4} + \frac{9}{5} + \frac{11}{6}$

23. $\sum_{j=1}^{3}\left(\sum_{k=1}^{j}(k-1)\right) = \sum_{k=1}^{1}(k-1) + \sum_{k=1}^{2}(k-1) + \sum_{k=1}^{3}(k-1)$

$= [(1-1)] + [(1-1) + (2-1)] + [(1-1) + (2-1) + (3-1)]$

$= 0 + [0 + 1] + [0 + 1 + 2]$

25. $-10, -14, -18, \ldots, -66$

$a_1 = -1$; d $= -4$

$a_n = a_1 + (n-1)d$

$-66 = -10 + (n-1)(-4)$

$14 = n - 1$

$n = 15$

$S_{15} = \frac{n}{2}(a_1 + a_n) = \frac{15}{2}(-10 + (-66)) = -570$

27. $a_1 = -\frac{3}{4}$, d $= \frac{1}{4}$; find S_{32}

$S_n = \frac{n}{2}[2a_1 + (n-1)\text{d}]$

$= \frac{32}{2}[2(-\frac{3}{4}) + 31(\frac{1}{4})] = 100$

29. $\frac{4}{3}, \frac{8}{9}, \ldots$; find S_5

$a_1 = \frac{4}{3}$; $r = \frac{\frac{8}{9}}{\frac{4}{3}} = \frac{2}{3}$.

$S_n = a_1\left(\frac{1 - r^n}{1 - r}\right)$

$S_5 = \frac{4}{3} \cdot \frac{1 - (\frac{2}{3})^5}{1 - (\frac{2}{3})} = \frac{4}{3} \cdot \frac{1 + \frac{32}{243}}{\frac{5}{3}}$

$= \frac{4}{3} \cdot \frac{3}{5} \cdot \frac{275}{243} = \frac{220}{243}$

31. $\sum_{k=1}^{10}(-3)^k$ $\quad$; $a_1 = -3, r = -3$, n = 10: $S_{10} = -3 \cdot \frac{1 - (-3)^{10}}{1 - (-3)}$

$= -\frac{3}{4}(1 - 59049) = 44286.$

33. $\sum_{i=1}^{\infty} 4(\frac{1}{2})^i$; $a_1 = 2, r = \frac{1}{2}$: $S = \frac{a_1}{1 - r} = \frac{2}{1 - \frac{1}{2}} = 4$

35. $0.3232\overline{32} = \frac{32}{100} + \frac{32}{100^2} + \frac{32}{100^3} + \ldots$;

$a_1 = \frac{32}{100}, r = \frac{1}{100}$: $S = \frac{\frac{32}{100}}{1 - \frac{1}{100}} = \frac{32}{100} \cdot \frac{100}{99} = \frac{32}{99}$

37. $0.322\overline{2} = 0.3 + (\frac{2}{100} + \frac{2}{1000} + \frac{2}{10000} + \ldots)$; $a_1 = \frac{2}{100} = \frac{1}{50}$,

$r = \frac{1}{10}$: $S = \frac{\frac{1}{50}}{1 - \frac{1}{10}} = \frac{1}{50} \cdot \frac{10}{9} = \frac{1}{45}$. Thus $0.322\overline{2} = \frac{3}{10} + \frac{1}{45} = \frac{29}{90}$

39. Assume the initial amount of bacteria is 1. After one hour the amount will be 115% of this, or 1.15; this is a_1; $r = 1.15$, because each generation is 115% of the precious one. We want $a_n = 2$. $a_n = a_1r^{n-1}$, so $2 = 1.15 \cdot 1.15^{n-1}$, $1.15^n = 2$

Trial and error shows that $1.15^4 \approx 1.75$, and $1.15^5 \approx 2.01$. Thus $n \approx 5$. It will take about 5 hours for the bacteria population to double.

41. $\binom{21}{18} = \frac{21!}{18! \cdot 3!} = \frac{21 \cdot 20 \cdot 19}{3 \cdot 2} = 1330$

43. $\binom{4}{0}(2x)^{4-0}(-y)^0 + \binom{4}{1}(2x)^{4-1}(-y)^1 + \binom{4}{2}(2x)^{4-2}(-y)^2$

$+ \binom{4}{3}(2x)^{4-3}(-y)^3 + \binom{4}{4}(2x)^{4-4}(-y)^4$

$1 \cdot 2^4x^4 + 4 \cdot 2^3x^3(-y) + 6 \cdot 2^2x^2y^2 + 4 \cdot 2x(-y^3) + 1 \cdot y^4$

$16x^4 - 32x^3y + 24x^2y^2 - 8xy^3 + y^4$

45. The 14'th term is where $i = 13$: $\binom{n}{i}x^{n-i}y^i$

$= \binom{16}{13}(5a^2)^{16-13}(b^5)^{13} = 560(5a^2)^3(b^{65}) = 560(5^3)a^6a^2b^{65} =$ 70,000a^6b^{65}

47. $\sum_{i=1}^{10}(4i^2 - 1) = \sum_{i=1}^{10} 4i^2 - \sum_{i=1}^{10} 1$

$= 4\sum_{i=1}^{10} i^2 - 10(1) = 4 \cdot \frac{10(11)(21)}{6} - 10 = 1530$

49. $\sum_{i=1}^{k} [i^2 - i + 1] = \sum_{i=1}^{k} i^2 - \sum_{i=1}^{k} i + \sum_{i=1}^{k} 1$

$= \frac{k(k+1)(2k+1)}{6} - \frac{k(k+1)}{2} + k(1)$

$= \frac{2k^3 + 3k^2 + k}{6} - \frac{3k^2 + 3k}{6} + \frac{6k}{6} = \frac{1}{3}(k^3 + 2k)$

51. $4 + 7 + 10 + ... + (3n + 1) = \frac{n(3n+5)}{2}$

Case $n = 1$: $3(1) + 1 = \frac{1(3(1)+5)}{2}$; $4 = 4$ ✔

Case $n = k$: $4 + 7 + 10 + ... + (3k + 1) = \frac{k(3k+5)}{2}$ (Assume true up to some k).

Case $n = k + 1$: $4 + 7 + 10 + ... + (3(k + 1) + 1)$

$= \frac{(k+1)(3(k+1)+5)}{2} = \frac{(k+1)(3k+8)}{2}$

Goal statement

Proof for $n = k$:

$4 + 7 + 10 + ... + (3k + 1) = \frac{k(3k+5)}{2}$ TRUE

$4 + 7 + 10 + ... + (3k + 1) + (3(k + 1) + 1)$

$= \frac{k(3k+5)}{2} + (3(k + 1) + 1)$ TRUE

because we have simply added the same amount to both sides. The left side is the left side of the goal statement. We now show that the right side is the right side of the goal statement.

$= \frac{k(3k+5)}{2} + \frac{2(3k+4)}{2} = \frac{3k^2 + 11k + 8}{2}$

$= \frac{(k+1)(3k+8)}{2}$ Right side of goal statement. ✔

53. Show that $n^3 - n$ is divisible by 3 for any natural number n.
Case $n = 1$: $1^3 - 1 = 0$, which is divisible by 3: $0 = 3 \cdot 0$.
Case $n = k$: Assume $k^3 - k = 3d$ some natural number d.
Case $n = k + 1$: $(k + 1)^3 - (k + 1) = (k + 1)[(k + 1)^2 - 1]$
$= (k + 1)(k^2 + 2k) = k^3 + 3k^2 + 2k = k^3 - k + 3k^2 + 3k$
(Add $-k$ to obtain $k^3 - k$.)
$= 3d + 3k^2 + 3k = 3(d + k^2 + k)$. The right side is divisible by 3; therefore the left side is divisible by 3. ✔

55. $4 \cdot 6 = 24$

57. 3 choices on question 1 • 3 choices on question 2 • … • 3 choices on question 8 = $3^8 = 6561$.

59. ${}_{10}P_2 = 10 \cdot 9 = 90$

61. 12 ways to choose a president • 11 ways to choose a vice-president = 132.

63. $\frac{5!}{2!2!} = 30$ Permutations of 5 things, but exclude the permutations of the 2 "a"s and the 2 "m"s.

65. ${}_nC_k = \frac{n!}{k!(n-k)!}$

Same

${}_nC_{n-k} = \frac{n!}{(n-k)![n-(n-k)]!} = \frac{n!}{(n-k)!k!}$

67. From 6 pigments, choose 2: ${}_6C_2 = 15$.

69. From 4 aces, choose 2. After this choice, from 4 kings choose 3. ${}_4C_2 \cdot {}_4C_1 = 6 \cdot 4 = 24$

71. a. $8! = 40{,}320$.

b. How many ways can 4 females and 4 males sit in a row if females and males must alternate?

F M F M F M F M

If a female is chosen first: $4 \cdot 4 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1 = 4!\,4!$

M F M F M F M F

If a male is chosen first: $4 \cdot 4 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1 = 4!\,4!$

Total is $2 \cdot 4! \cdot 4! = 1152$.

c. $\frac{\text{Permutations of 8 people}}{\text{Permutations of 4 females} \cdot \text{Permutations of 4 males}} = \frac{8!}{4!\,4!} = 70$

73. Selecting from the set of digits { 1, 2, 3, 4, 5, 6 } (repeat selections are allowed) how many of the following are possible?

a. 4-digit numbers. $6 \cdot 6 \cdot 6 \cdot 6 = 1296$

b. 4-digit odd numbers. $6 \cdot 6 \cdot 6 \cdot 3 = 648$ (The last digit must be from { 1, 3, 5 },

c. 3-digit numbers where the first digit must be even. $3 \cdot 6 \cdot 6 = 108$

d 3-digit numbers using only even digits. $3 \cdot 3 \cdot 3 = 27$

75. If each team had to play every other team once, the number would be ${}_8C_2 = 28$. Double this value to obtain 56 total games to be played.

77. a. ${}_{10}C_3 \cdot {}_{12}C_3 = 26400$.

b. Number of all female committees plus number of all male committees = ${}_{12}C_6 + {}_{10}C_6 = 1134$

c. ${}_{22}C_6 = 74{,}613$

A coin is tossed 4 times.
S = { HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT, THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT }.

79. one head and three tails $\frac{4}{16} = \frac{1}{4}$

81. a five. There are 4 "5"s out of 52 cards: $\frac{4}{52} = \frac{1}{13}$

83. a card from 4 through 10, inclusive. This is 7 cards from each suit, or 28 cards: $\frac{28}{52} = \frac{7}{13}$

85. P(diamond or jack) = P(diamond) + P(jack) − P(jack of diamonds) = $\frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$

87. a card that is not a club: P(not club) = 1 − P(club) = $1 - \frac{13}{52} = \frac{3}{4}$

89. red or blue? $\frac{4}{14} + \frac{4}{14} = \frac{4}{7}$ (Mutually exclusive events)

91. S = 6 plants; n(S) = ${}_{12}C_6 = 924$. A = 2 diseased plants, 4 healthy plants; n(A) = ${}_4C_2 \cdot {}_8C_4 = 420$ P(A) = $\frac{n(A)}{n(S)} = \frac{420}{924} = \frac{5}{11}$

S = five cards drawn from 52, order not important.
n(S) = ${}_{52}C_5 = 2{,}598{,}960$.

93. A = All five cards are clubs.
n(A) = ${}_{13}C_5 = 1287$; P(A) = $\frac{1287}{2598960} \approx 0.00050$

95. A = None of the cards are red.
Same as all five cards are black. This probability is the same as for all five cards are red, which is 0.0253 from the last problem.

97. $\frac{1}{{}_{49}C_6} = \frac{1}{13983816} \approx 0.000\,000\,07$

99. $a_n = \begin{cases} 2 \text{ if } n = 0 \\ a_{n-1} + 6 \text{ if } n > 0 \end{cases}$ $a_0 = 2, a_1 = 8, a_2 = 14, a_3 = 20,$ $a_4 = 26$. Each term is found by adding 6 to the previous term, so this is an arithmetic sequence.

$a_n = 2 + (n)(6)$ Remember, n starts at 0.

$a_n = 6n + 2$

101. $a_n = \begin{cases} 2 \text{ if } n = 0,\ 3 \text{ if } n = 1 \\ 2a_{n-1} + a_{n-2} \text{ if } n>1 \end{cases}$

First five terms: **2**, **3**, $2(3) + 2 =$ **8**, $2(8) + 3) =$ **19**, $2(19) + 8 =$ **46**

To find the general term:

$$a_n = 2a_{n-1} + a_{n-2}$$
$$a_n - 2a_{n-1} - a_{n-2} = 0$$
$$x^2 - 2x - 1 = 0$$
$$x = 1 + \sqrt{2} \text{ or } 1 - \sqrt{2}.$$
$$a_n = A(1+\sqrt{2})^n + B(1-\sqrt{2})^n$$

To find A and B:

$n = 0$: $2 = A + B$, so $B = 2 - A$

$n = 1$: $3 = A(1+\sqrt{2}) + B(1-\sqrt{2})$

$$3 = A(1+\sqrt{2}) + (2-A)(1-\sqrt{2})$$
$$1 + 2\sqrt{2} = 2A\sqrt{2}$$
$$A = \frac{1+2\sqrt{2}}{2\sqrt{2}}$$

$$B = 2 - \frac{1+2\sqrt{2}}{2\sqrt{2}} = \frac{4\sqrt{2}}{2\sqrt{2}} - \frac{1+2\sqrt{2}}{2\sqrt{2}} = \frac{2\sqrt{2}-1}{2\sqrt{2}}$$

$$a_n = \frac{1+2\sqrt{2}}{2\sqrt{2}}(1+\sqrt{2})^n - \frac{1-2\sqrt{2}}{2\sqrt{2}}(1-\sqrt{2})^n$$

103. $a_n = \begin{cases} 1 \text{ if } n = 0,3 \text{ if } n = 1 \\ 4a_{n-1} - 4a_{n-2} \text{ if } n>1 \end{cases}$ First five terms: **1**, **3**, $4(3) - 4(1) =$ **8**, $4(8) - 4(3) =$ **20**, $4(20) - 4(8) =$ **48**

To find the general term:

$$a_n = 4a_{n-1} - 4a_{n-2}$$
$$a_n - 4a_{n-1} + 4a_{n-2} = 0$$
$$x^2 - 4x + 4 = 0$$
$$(x-2)^2 = 0$$
$$x = 2 \text{ (multiplicity 2)}$$
$$a_n = A(2^n) + Bn(2^n)$$

To find A and B:

$n = 0$: $1 = A$

$n = 1$: $3 = 2A + 2B$

$$3 = 2 + 2B$$
$$\tfrac{1}{2} = B$$

Thus $a_n = 2^n + \frac{n}{2}2^n$ or $(\frac{n}{2} + 1)2^n$.

Chapter 12 Test

1. $a_n = (-1)^{n+1}(-n+3)$

$(-1)^2(-1+3)$, $(-1)^3(-2+3)$, $(-1)^4(-3+3)$, $(-1)^5(-4+3)$

$2, -1, 0, 1$

3. $6, 10, 14, 18, \ldots$

$6, 6+4, 6+8, 6+12, \ldots$

$6 + (n-1)(4)$

$4n + 2$

5. $a_1 = 103$, d = 3

$a_n = a_1 + (n-1)$d

$184 = 103 + (n-1)(3)$

$27 = n - 1$

$28 = n$

7. Let $A = 2, 6, 10, 14, \ldots$ be an arithmetic sequence.

Then if $b_n = 3a_n$, $B = 6, 18, 30, 42, \ldots$ which seems to be an arithmetic sequence. Thus, we shall try to show that B is always arithmetic.

$$b_n = 3a_n$$
$$= 3(a_1 + (n-1)d_a)$$
$$= 3a_1 + (n-1)(3d_a)$$
$$= b_1 + (n-1)d_b$$

Thus B is an arithmetic sequence, where $b_1 = 3a_1$, and $d_b = 3d_a$.

9. $\frac{a_5}{a_3} = \frac{25}{200} = \frac{1}{4} = \frac{a_1r^4}{a_1r^2} = r^2$. Thus $r^2 = \frac{1}{4}$, so $r = \pm\frac{1}{2}$.

$a_7 = a_5r^2$ $\quad (a_6 = a_5r$, so $a_7 = a_5r^2)$

$a_7 = 25 \cdot \frac{1}{4} = 6\frac{1}{4}$

11.

First	**Second**	Third	**Fourth**
$\frac{2}{3}(36)$	$\frac{2}{3}$**(24)**	$\frac{2}{3}(16)$	$\frac{2}{3} \cdot \frac{32}{3}$
24	**16 (a)**	$\frac{32}{3}$	$\frac{64}{9}$ **(b)**

The heights of the bounces are a geometric series with a_1 the height of the first bounce and $r = \frac{2}{3}$. Thus, $a_n = 24(\frac{2}{3})^{n-1}$ ft.

13. $(-1)^1(1-3)^2 + (-1)^2(2-3)^2 + (-1)^3(3-3)^2 + (-1)^4(4-3)^2 + (-1)^5(5-3)^2$; $-4 + 1 + 0 + 1 - 4$

15. $a_1 = -20$, $d = 2$

To find n: $22 = a_1 + (n-1)d = -20 + (n-1)(2)$

$$22 = n$$

$S_n = \frac{n}{2}(a_1 + a_n)$; $S_{22} = 11(-20 + 22) = 22$.

17. $S_5 = \frac{1}{3}\left(\frac{1-6^5}{1-6}\right) = \frac{1}{3} \cdot \frac{-7775}{-5} = 518\frac{1}{3}$

19. $\sum_{k=1}^{8} 512(-\frac{1}{2})^k$; $a_1 = -256$, $r = -\frac{1}{2}$; $S_8 = -256\left(\frac{1-(-\frac{1}{2})^8}{1-(-\frac{1}{2})}\right) =$

$-256 \cdot \frac{2}{3}(1 - \frac{1}{256}) = \frac{256 \cdot 2}{3} \cdot \frac{255}{256} = -170$

21. $\sum_{i=1}^{\infty} 3^i$ Since $r = 3 \geq 1$, the sum is not defined.

23. $0.2727\overline{27} = \frac{27}{100} + \frac{27}{100^2} + \frac{27}{100^3} + \ldots$

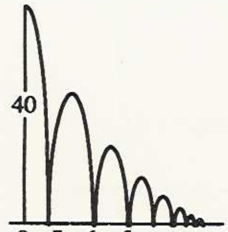
Infinite geometric series, $a_1 = \frac{27}{100}$, $r = \frac{1}{100}$:

$$S = \frac{\frac{27}{100}}{1 - \frac{1}{100}} = \frac{27}{100} \cdot \frac{100}{99} = \frac{3}{11}$$

25. distance = 40 + 2[40(0.75) + 40(0.75^2) + 40(0.75)^3 + ...]
The expression in brackets is an infinite geometric series.

$$S = \frac{40(0.75)}{1 - 0.75} = 120$$

Thus distance = 40 + 2(120) = 280 meters.



27. $\binom{8}{4} = \frac{8!}{4!\,(8-4)!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4!}{4! \cdot 4!} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2} = 70$

29. $(x^2 - 3y)^4 = \sum_{i=0}^{4} \binom{4}{i} (x^2)^{4-i}(-3y)^i$

$= \binom{4}{0}(x^2)^{4-0}(-3y)^0 + \binom{4}{1}(x^2)^{4-1}(-3y)^1 + \binom{4}{2}(x^2)^{4-2}(-3y)^2$

$+ \binom{4}{3}(x^2)^{4-3}(-3y)^3 + \binom{4}{4}(x^2)^{4-4}(-3y)^4$

$= x^8 + 4x^6(-3y) + 6x^4(3y)^2 + 4x^2(-3y)^3 + (-3y)^4$

$= x^8 - 12x^6y + 54x^4y^2 - 108x^2y^3 + 81y^4$

31. $\sum_{i=1}^{14} (i+3)^2 = \sum_{i=1}^{14} (i^2 + 6i + 9) = \sum_{i=1}^{14} i^2 + 6\sum_{i=1}^{14} i + \sum_{i=1}^{14} 9$

$= \frac{14(15)(29)}{6} + 6\left(\frac{14(15)}{2}\right) + 14(9) = 1771$

33. $\sum_{i=1}^{k} (2i + 1) = 2\sum_{i=1}^{k} i + \sum_{i=1}^{k} 1 = 2\left(\frac{k(k+1)}{2}\right) + k(1) = k^2 + 2k$

35. $5 + 9 + 13 + \ldots + (4n + 1) = 2n^2 + 3n$

Case $n = 1$: $(4(1) + 1) = 2(1^2) + 3(1)$; $5 = 5$ ✔

Case $n = k$: $5 + 9 + 13 + \ldots + (4k + 1) = 2k^2 + 3k$

Case $n = k + 1$: $5 + 9 + 13 + \ldots + (4(k + 1) + 1)$
$= 2(k + 1)^2 + 3(k + 1)$ (GOAL)
$= 2k^2 + 7k + 5$ (Right member expanded.)

Proof for $n = k + 1$:
$5 + 9 + 13 + \ldots + (4k + 1) = 2k^2 + 3k$ TRUE
$5 + 9 + 13 + \ldots + (4k + 1) + (4(k + 1) + 1)$
$= 2k^2 + 3k + (4(k + 1) + 1)$ True because we have added the same amount to both sides. The left side is now the left side of the goal statement.
$= 2k^2 + 7k + 5$

This last expression is the right side of the goal statement. ✔

37. Show that $n^2 + 7n + 12$ is divisible by 2 for any natural number n.

$n = 1$: $1^2 + 7 + 12 = 20$, which divisible by 2. ✔

Assume true for $n = k$; that is, $k^2 + 7k + 12 = 2m$ for some integer m.

For $n = k + 1$: $(k + 1)^2 + 7(k + 1) + 12$
$= k^2 + 9k + 20$
$= k^2 + 7k + 12 + 2k + 8$
$= 2m + 2k + 8$
$= 2(m + k + 8)$. ✔

39. $5 \cdot 4 \cdot 6 = 120$

41. $13 \cdot 12 \cdot 13 \cdot 12 = 24336$

43. $15 \cdot 14 = 210$

45. (a) $26! \approx 4.0329 \times 10^{26}$

(b) $\frac{4.0329 \times 10^{26}}{10} = 4.0329 \times 10^{25}$ seconds required. To convert to years, compute:

$\frac{4.0329 \times 10^{25} \text{ s}}{1} \times \frac{1 \text{ min}}{60 \text{ s}} \times \frac{1 \text{ h}}{60 \text{ min}} \times \frac{1 \text{ day}}{24 \text{ h}} \times \frac{1 \text{ year}}{365 \text{ days}} =$

$\frac{4.0329 \times 10^{25}}{60 \times 60 \times 24 \times 365} \approx 1.28 \times 10^{18}$ years. (It is thought the universe is less than 20 billion years old; this is 20×10^9 years.)

47. $\frac{n!}{(n-2)!\,[n-(n-2)]!} = \frac{n(n-1)(n-2)!}{(n-2)!\,2!} = \frac{n(n-1)}{2}$

49. ${}_{23}C_2 = 253$

51. ${}_{20}C_9 = 167960$

53. ${}_{20}C_9 \cdot 9! \approx 6.09 \times 10^{10}$

55. Number of six question tests x Number of ways to answer a six question test: ${}_{10}C_6 \cdot 4^6 = 860160$

57. ${}_{24}C_2 = 276$

59. S = {HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT, THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT}.
$\frac{6}{16} = \frac{3}{8}$

61. P(red or face) = P(red) + P(face) − P(red face card)
$= \frac{26}{52} + \frac{12}{52} - \frac{6}{52} = \frac{32}{52} = \frac{8}{13}$

63. P(not white) = 1 − P(white) = $1 - \frac{6}{22} = \frac{8}{11}$

65. $\frac{1}{{}_{20}C_2} = \frac{1}{190} \approx 0.005$.

67. $a_n = \begin{cases} 5 \text{ if } n = 0 \\ a_{n-1} + 2 \text{ if } n > 0 \end{cases}$: 5, 7, 9, 11, 13 ; $a_n = 5 + 2n$

69. $a_n = \begin{cases} 2 \text{ if } n = 0 \\ 4 \text{ if } n = 1 \\ 2a_{n-1} - a_{n-2} \text{ if } n > 1 \end{cases}$: 2, 4, 6, 8, 10, 12; $a_n = 2(n + 1)$


College Loan
CLC
Corporation
Need more money
for college expenses?
The CLC Private Loan℠ can get you up to
$40,000 a year for college-related expenses.
Here's why the CLC Private Loan℠ is a smart choice:
Approved borrowers are sent a check within four business days
Get $1000 - $40,000 each year
Interest rates as low as prime + 0% (8.66% APR)
Quick and easy approval process
No payments until after graduating or leaving school
CLICK HERE
or call 800.311.9615.
We are available 24 hours
a day, 7 days a week.